ON THE DUALITY OF TELEPORTATION AND DENSE CODING

arXiv:2302.14798

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Eric Chitambar

Quantum teleportation

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Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels

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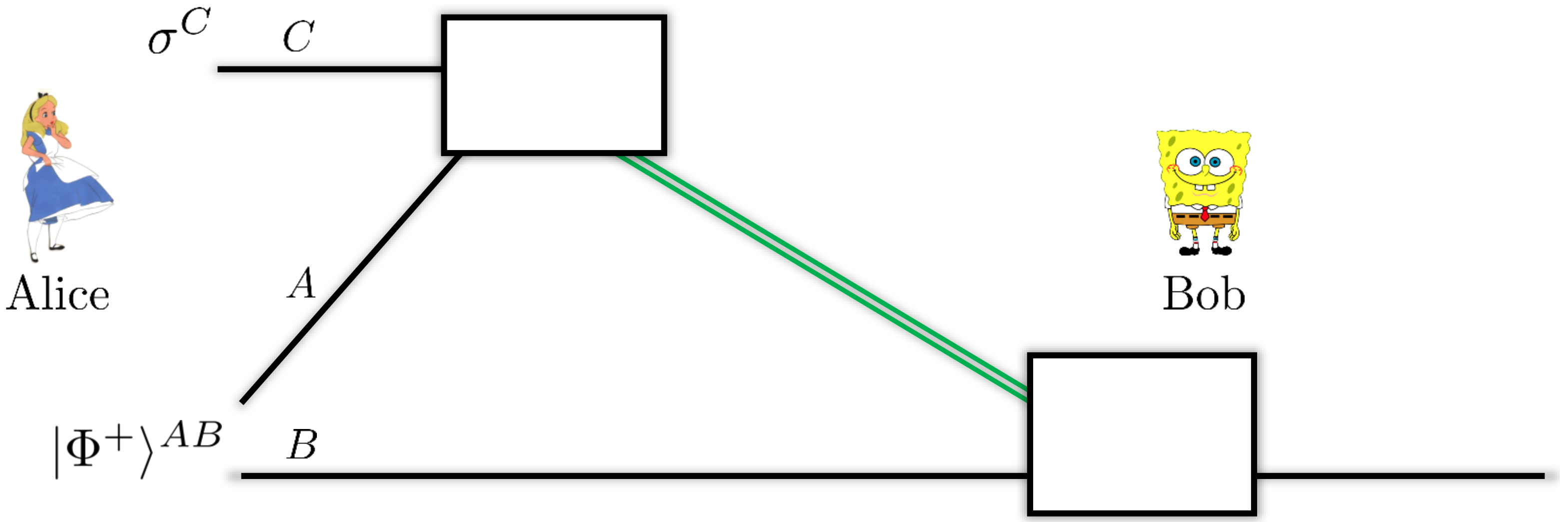
(Received 2 December 1992)

Quantum teleportation



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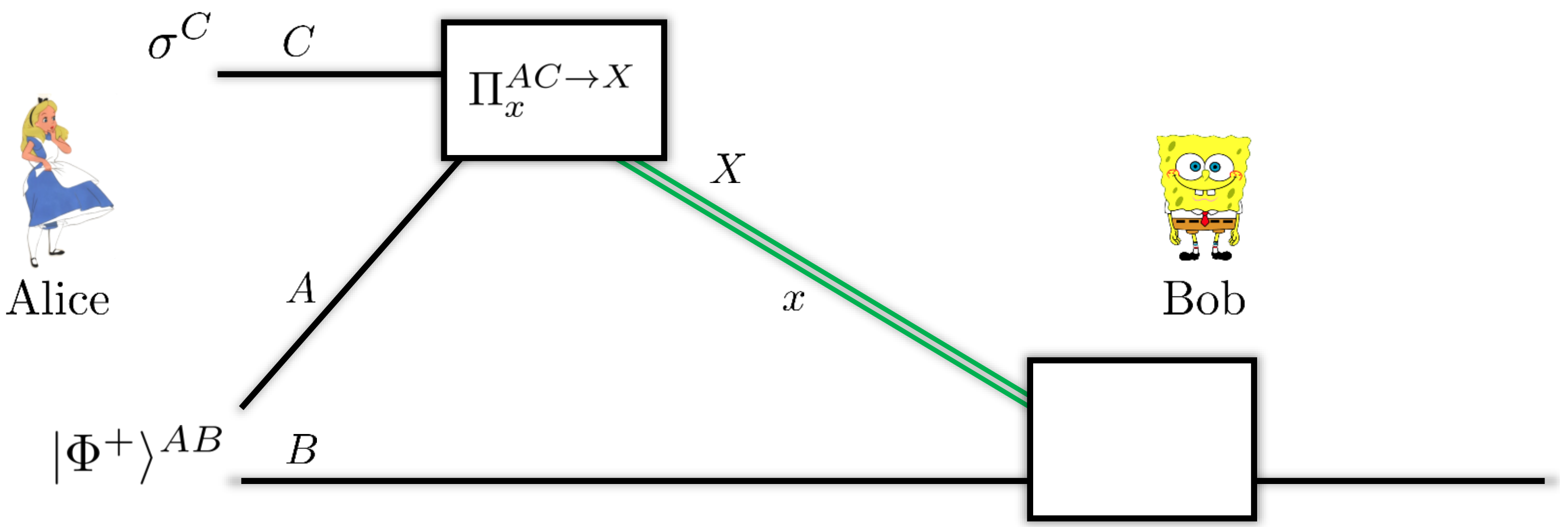

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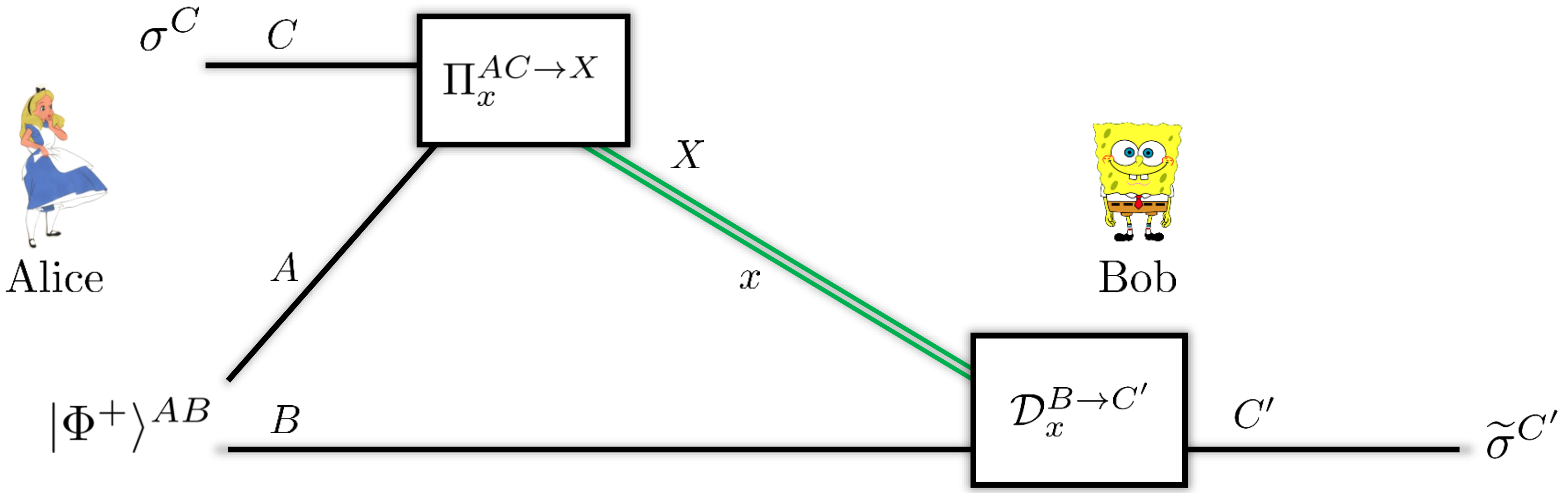

- Alice applies a qq-to-c encoder, which is a positive-operator valued-measure (POVM) $\{\Pi_x^{AC\to X}\}_x$

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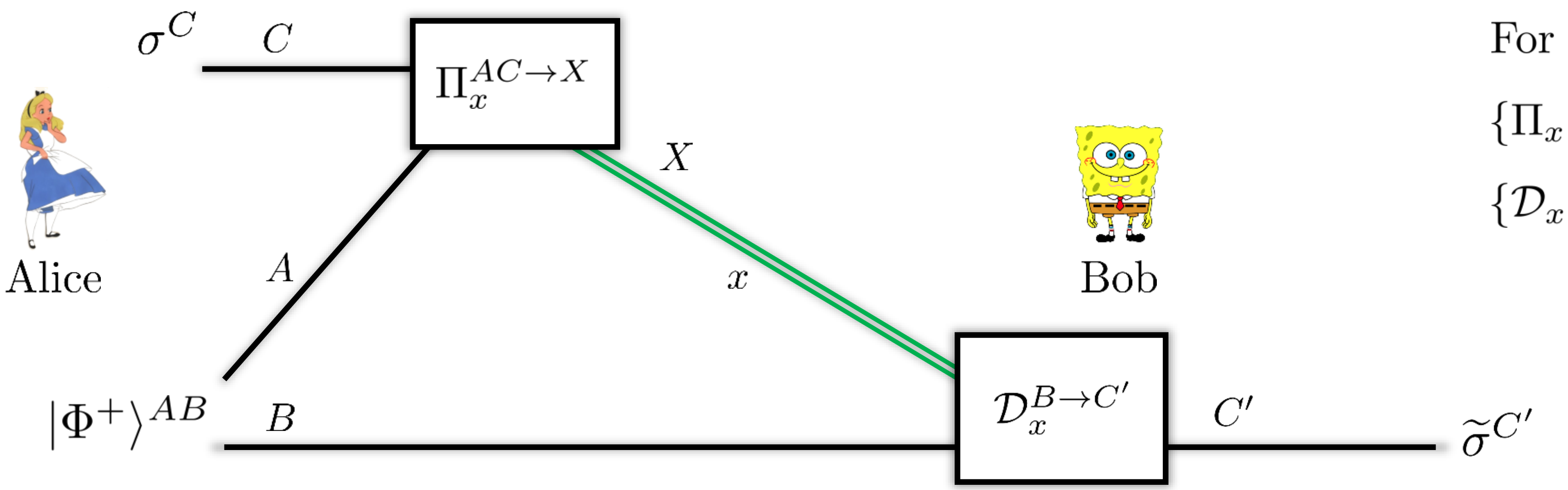

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$\{\Pi_x\}_x$ is a Bell-state measurement

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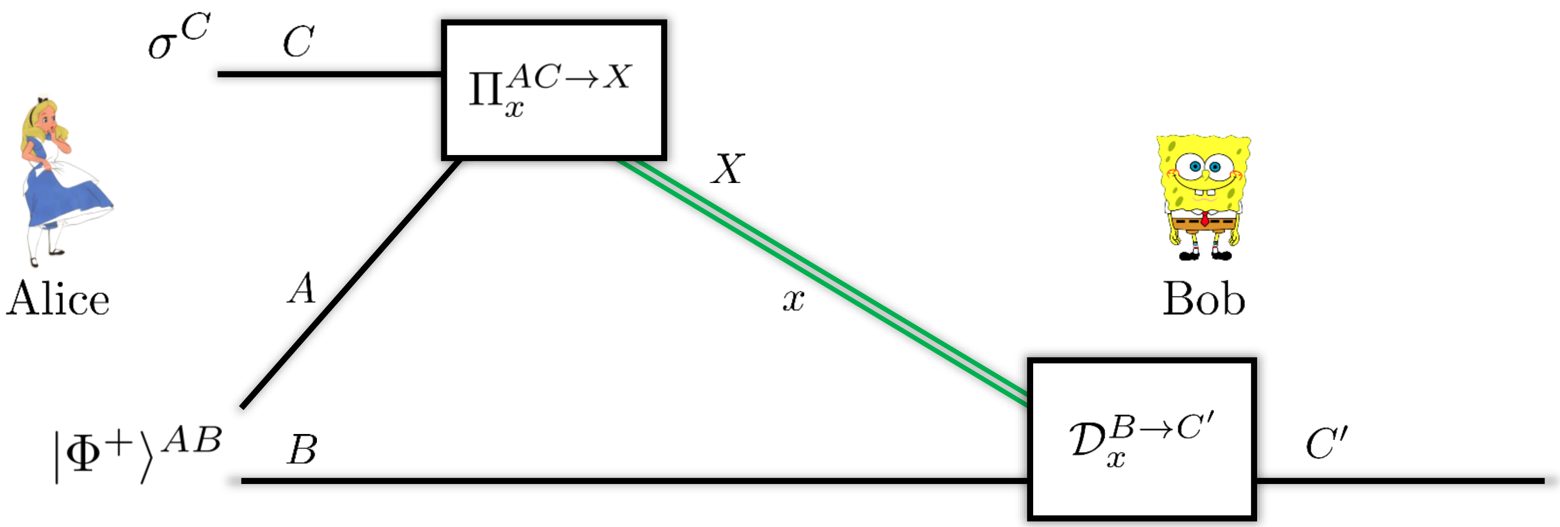

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$\Downarrow$

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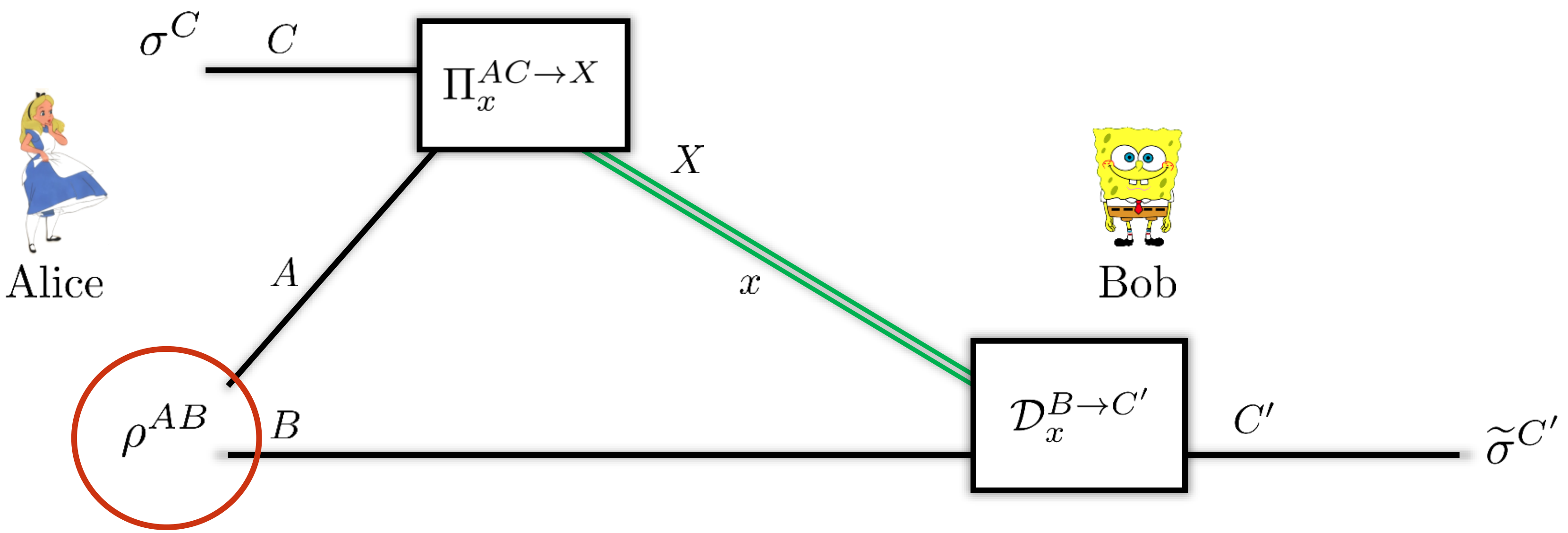

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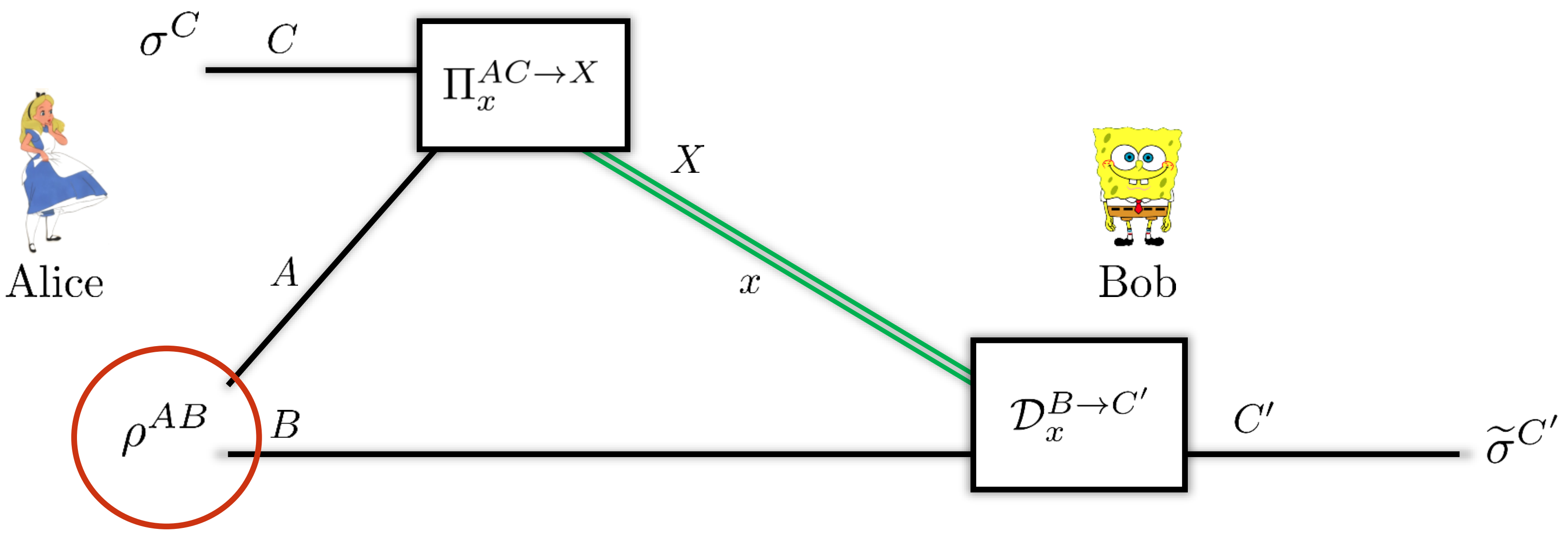

- Every bipartite state ρ^{AB}, encoding POVM $\{\Pi_x^{AC}\}$, and decoding maps $\{\mathcal{D}_x^{B\to C'}\}_x$ define a $|C|$-dimensional **one-way teleportation protocol** $(\rho, \{\Pi_x\}, \{\mathcal{D}_x\})$ for channel $\Lambda : C \to C'$,

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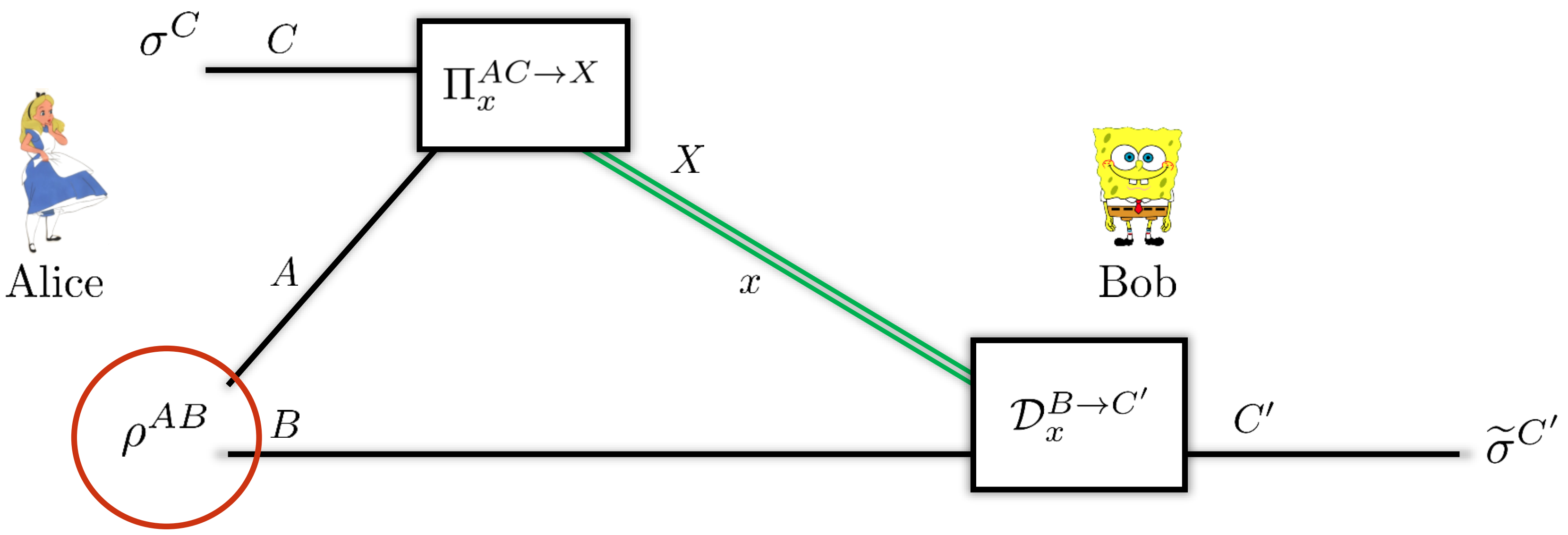

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- How good of a quantum communication resource is this channel?

How good are general teleportation channels?

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General teleportation channel, singlet fraction, and quasidistillation

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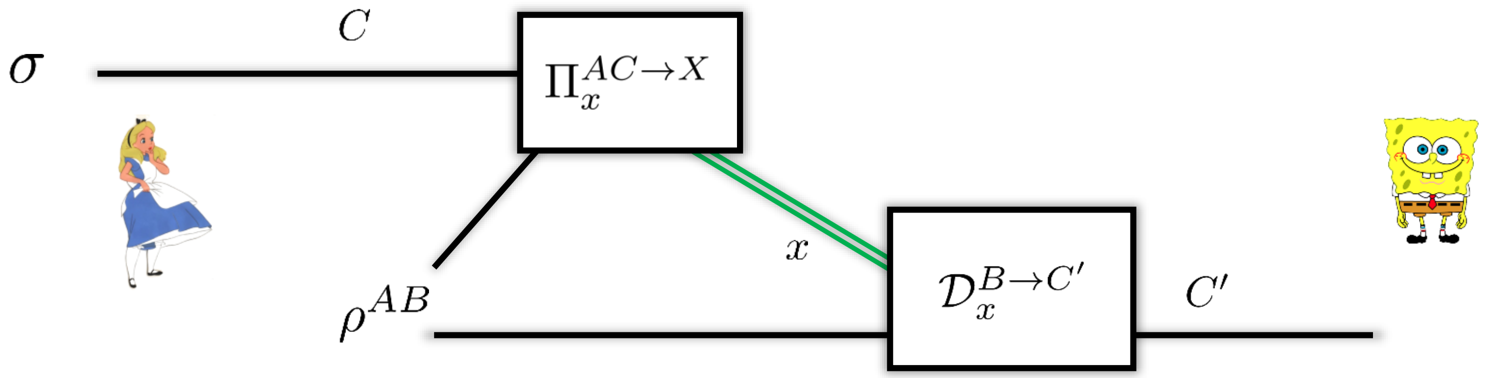

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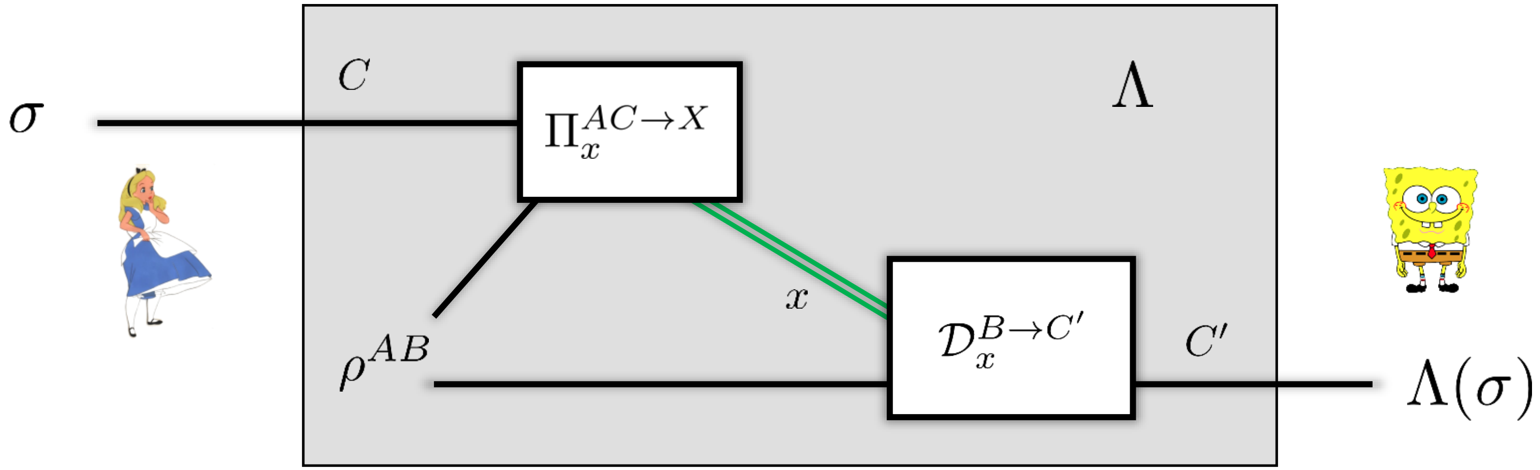

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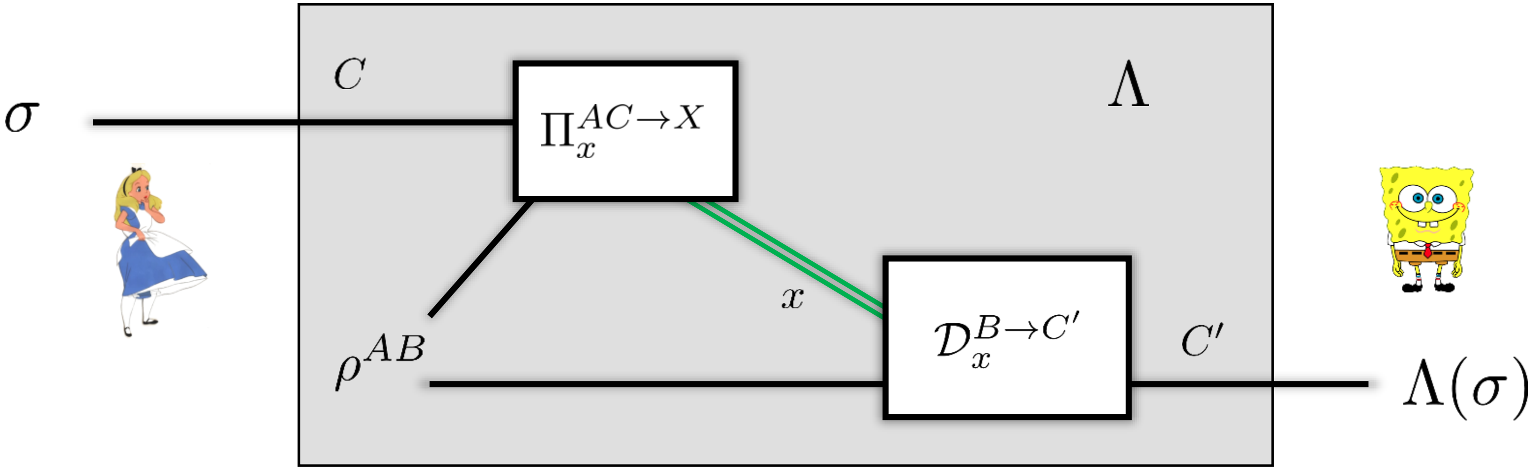

- The traditional measure used to assess the quality of a teleportation protocol is the **teleportation fidelity**:

$$f(\Lambda) = \int d\psi \langle\psi|\Lambda(\psi)|\psi\rangle.$$

Horodecki *et al.* PRA **60** 1888 (1999)

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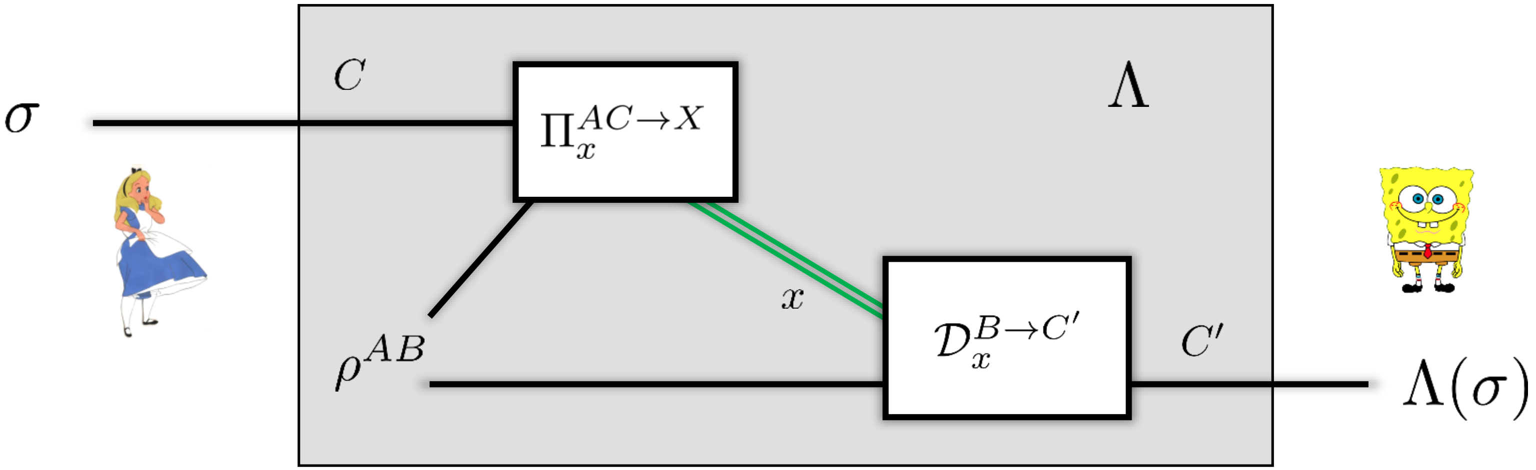

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- This is the fidelity of transmission averaged over all input pure states $|\psi\rangle$.

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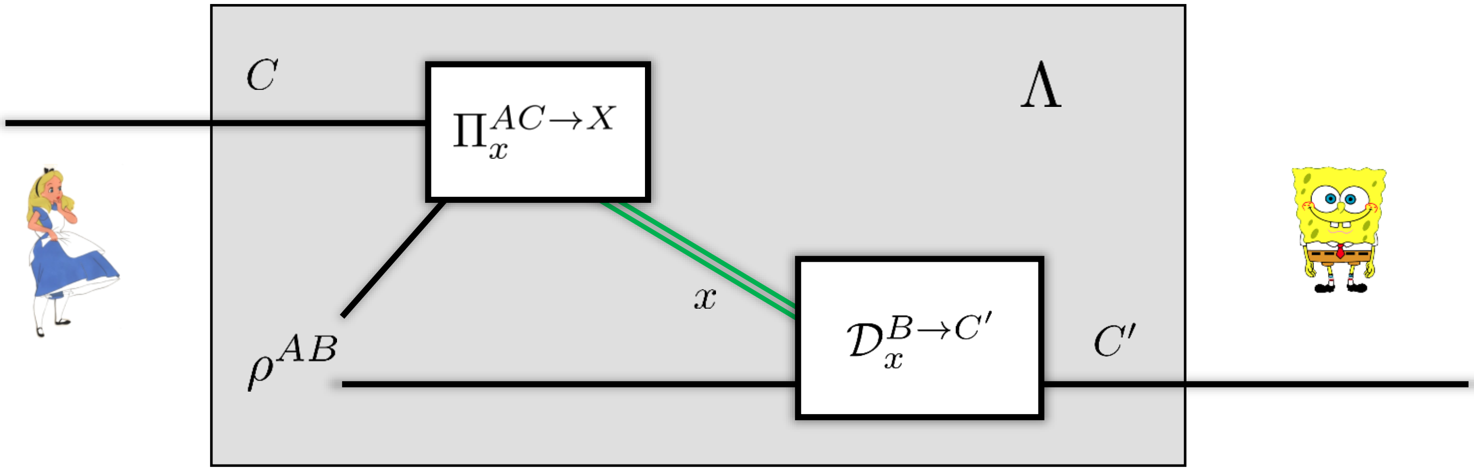

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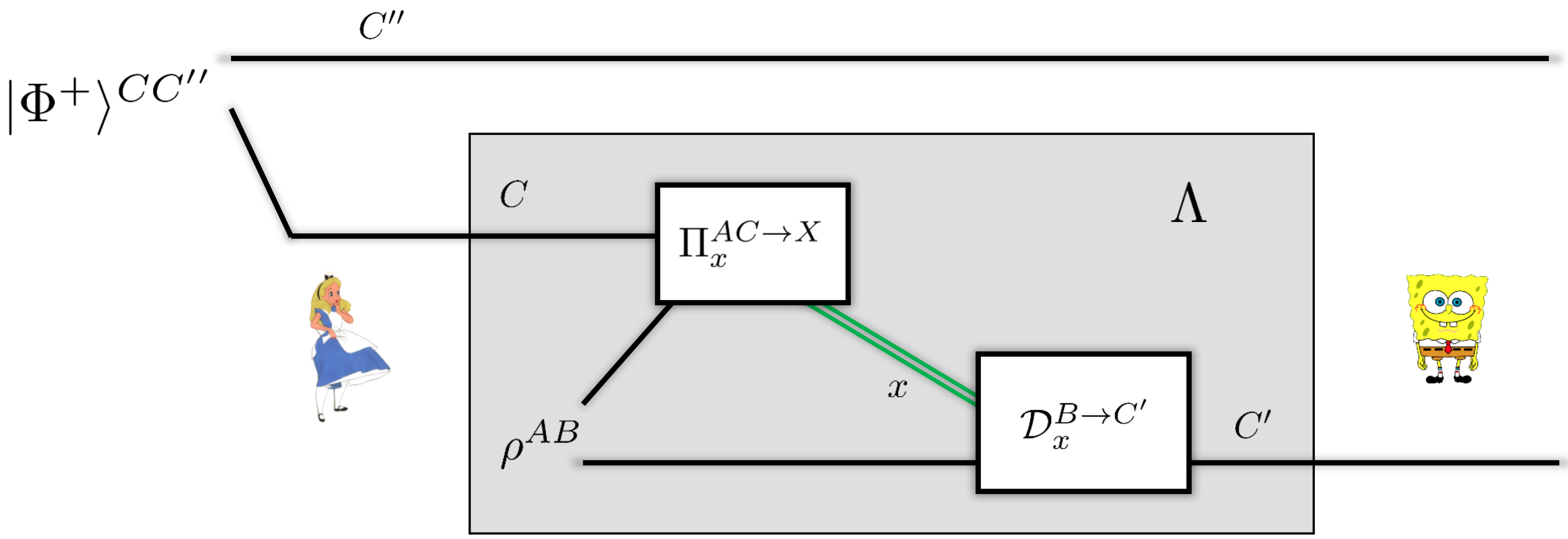

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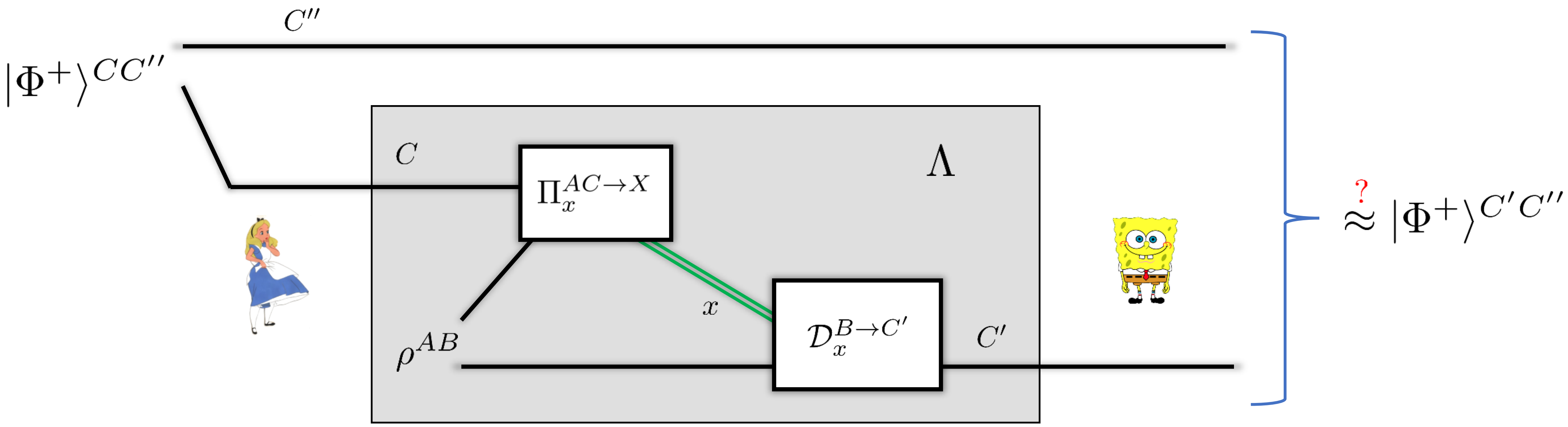

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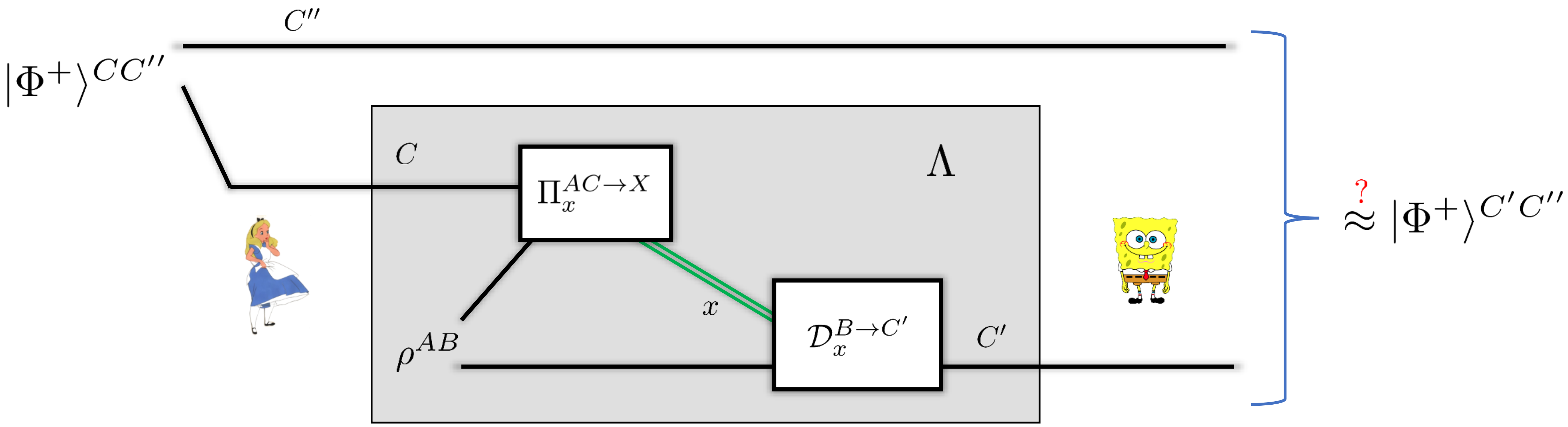

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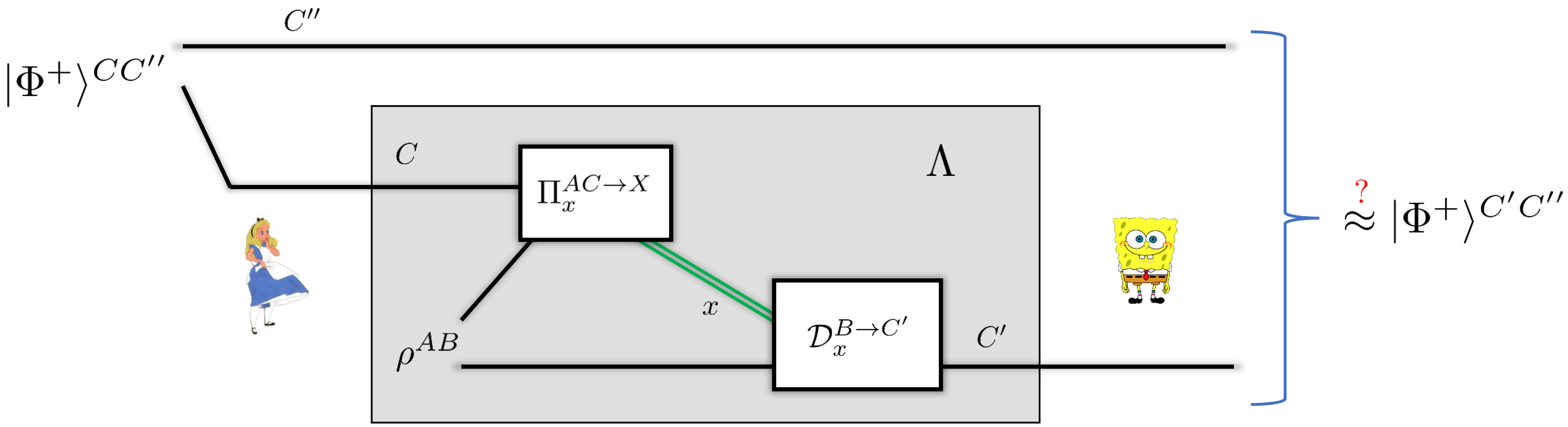

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Theorem [Horodecki *et al.* PRA '99]

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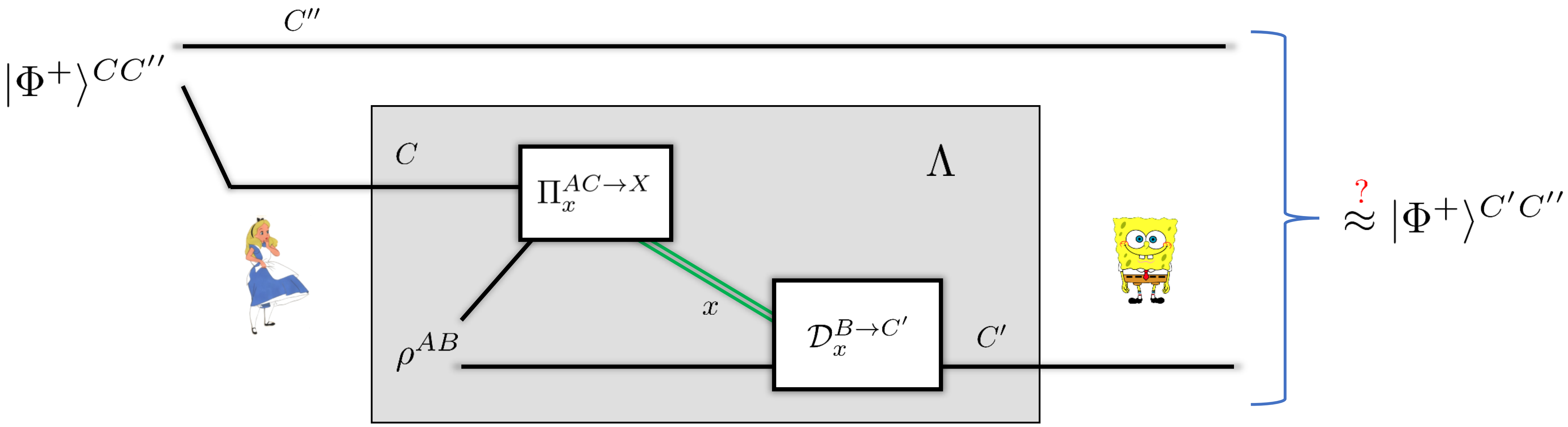

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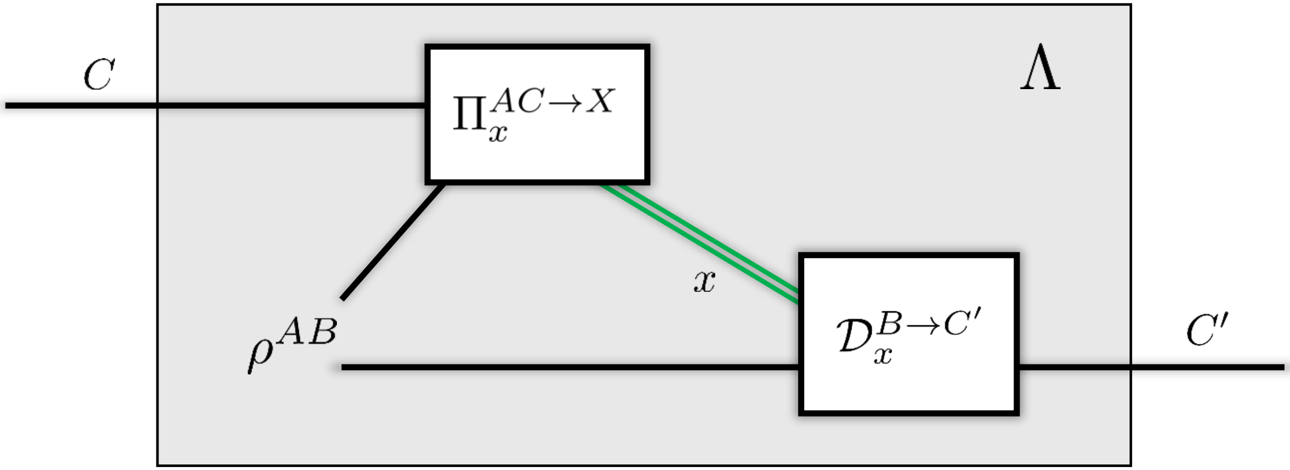
- **Our interest:** How else to interpret the teleportation entanglement fidelity $F(\Lambda)$?

General one-way teleportation protocols



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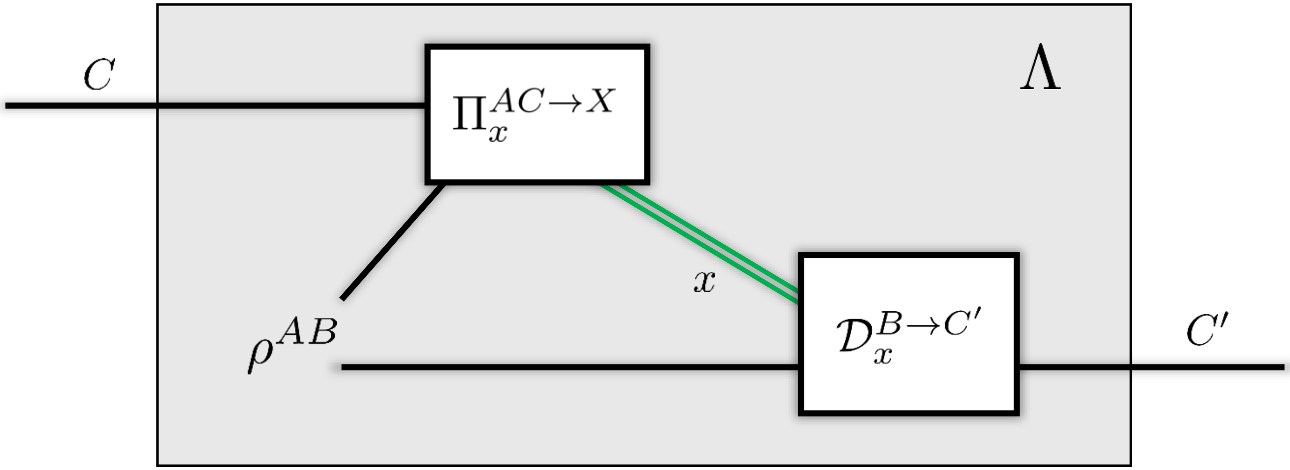

Chitambar, FL, arXiv:2302.14798

Lemma: Let $(\rho^{AB}, \{\Pi_x\}, \{\mathcal{D}_x\})$ define a one-way teleportation protocol. Then the teleportation entanglement fidelity is given by

$$F = \frac{N}{|C|^2}\left(\frac{1}{N}\sum_{x=1}^{N}\mathrm{Tr}[\Pi_x^{AC}\omega_x^{AC}]\right), \qquad \text{where } \omega_x = \mathrm{id}^A \otimes \mathcal{D}_x^{B\to C}(\rho^{AB}).$$

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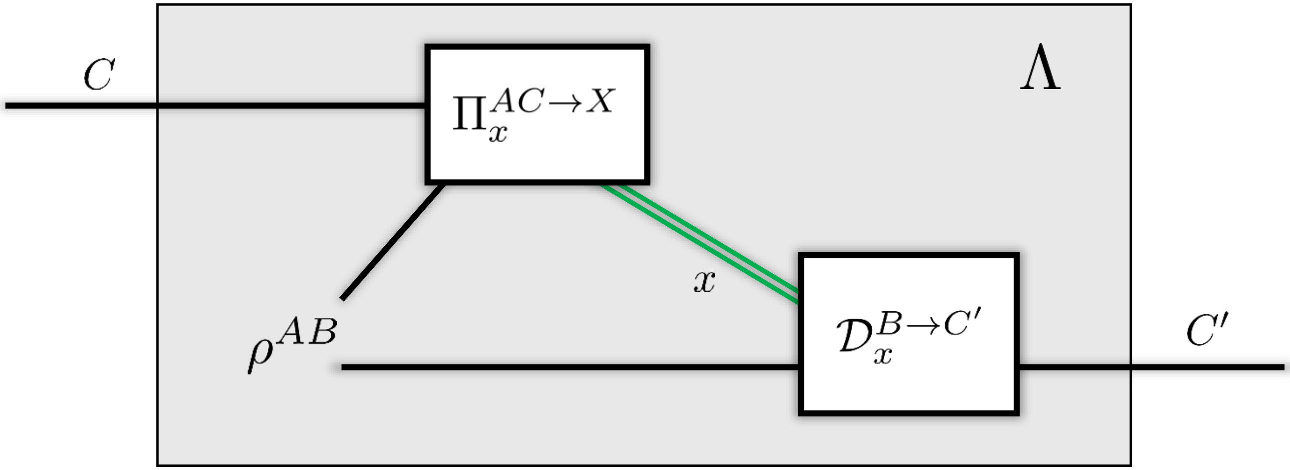

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Interpretation: The entanglement fidelity of every one-way teleportation protocol is equivalent to the success probability of a corresponding quantum state discrimination protocol.

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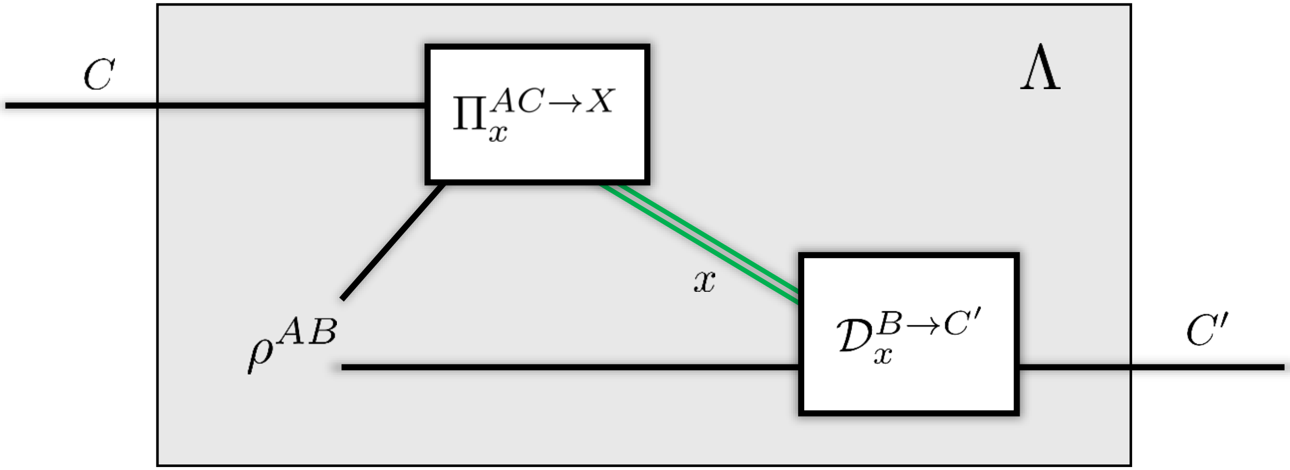

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This **generalizes** a similar formula for port-based teleportation protocols to **arbitrary teleportation protocols**.

Ishizaka, Hiroshima, PRL 101, 240501 (2008)

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$$\Rightarrow \quad F(\rho^{AB};|C|) \geq \frac{1}{|C|}.$$

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Fundamental question: When is this lower bound tight?

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Horodecki & Horodecki PRA **59** 4206 (1999)

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Corollary: Bound entangled states cannot exceed the (one-way) teleportation classical threshold.

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Theorem: A bipartite state ρ^{AB} is useful for $|C|$-dimensional teleportation (i.e., exceed the classical fidelity threshold $|C|^{-1}$)) iff there exists a channel $\mathcal{E}^{B\to C}$ such that $\omega^{AC} = \mathrm{id}^A \otimes \mathcal{E}^{B\to C}(\rho^{AB})$ violates the reduction criterion.

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- Compare with the classic result of the Horodeckis:

Theorem: A bipartite state ρ^{AB} is useful for $|C|$-dimensional teleportation (i.e., exceed the classical fidelity threshold $|C|^{-1}$)) iff there exists a one-way LOCC map $\mathcal{L}$ such that $\mathcal{L}(\rho)$ has a singlet fraction exceeding $|C|^{-1}$:

$$\langle \Phi^+ | \mathcal{L}(\rho) | \Phi^+ \rangle^{C\tilde{C}} > \frac{1}{|C|}.$$

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- Compare with the classic result of the Horodeckis:

Theorem: A bipartite state ρ^{AB} is useful for $|C|$-dimensional teleportation (i.e., exceed the classical fidelity threshold $|C|^{-1}$)) iff there exists a one-way LOCC map $\mathcal{L}$ such that $\mathcal{L}(\rho)$ has a singlet fraction exceeding $|C|^{-1}$:

$$\langle\Phi^+|\mathcal{L}(\rho)|\Phi^+\rangle^{C\tilde{C}} > \frac{1}{|C|}.$$

Horodecki *et al.* PRA **60** 1888 (1999)

- Our work simplifies the condition for non-classical teleportation fidelity from an optimization over all one-way LOCC maps to an optimization over just local maps.

Example: qutrit Werner states

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- Consider the $3 \otimes 3$ family of Werner states:

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⇨ Even states satisfying the reduction criterion can be useful for teleportation.

From teleportation to dense coding

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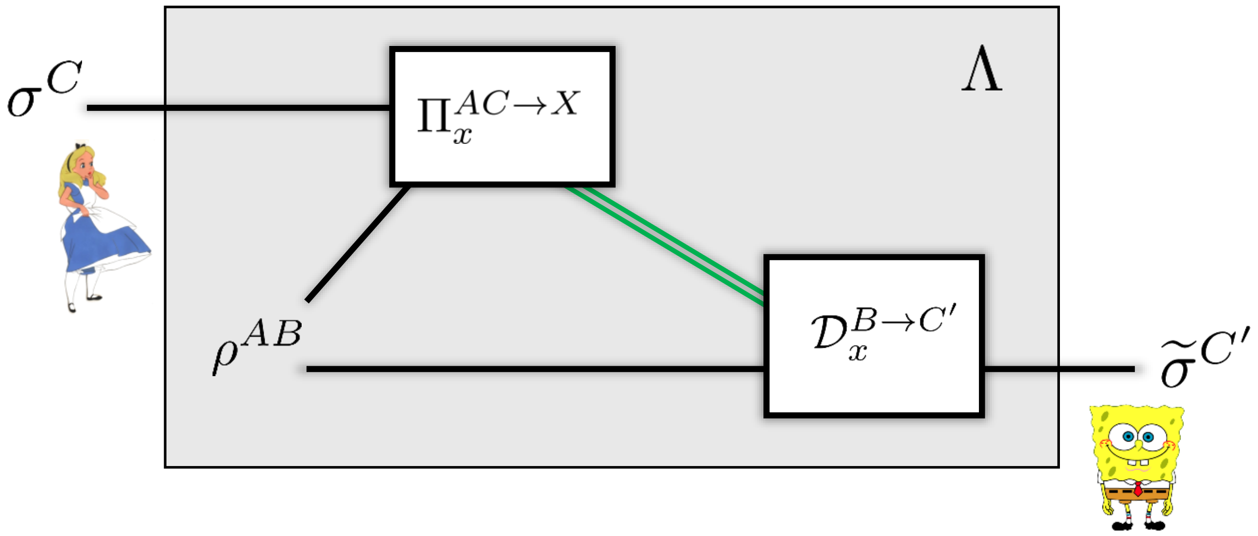
ρ^{AB}	$\{\Pi_x\}_{x=1}^N$	$\{\mathcal{D}_x\}_{x=1}^N$
Alice-to-Bob teleportation	Quantum encoder	Quantum decoder



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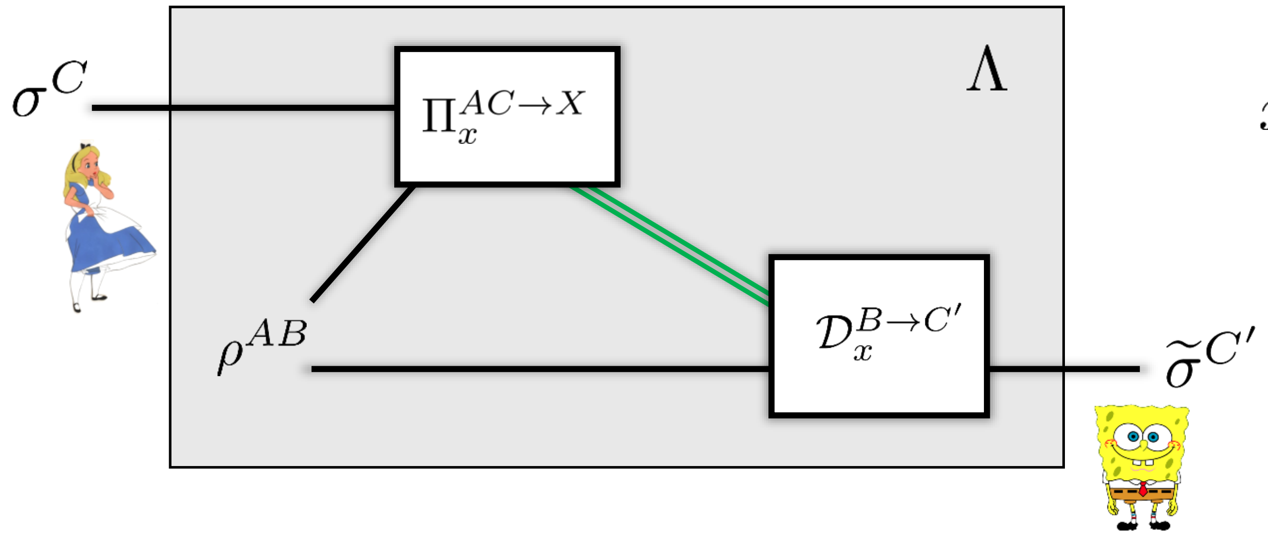

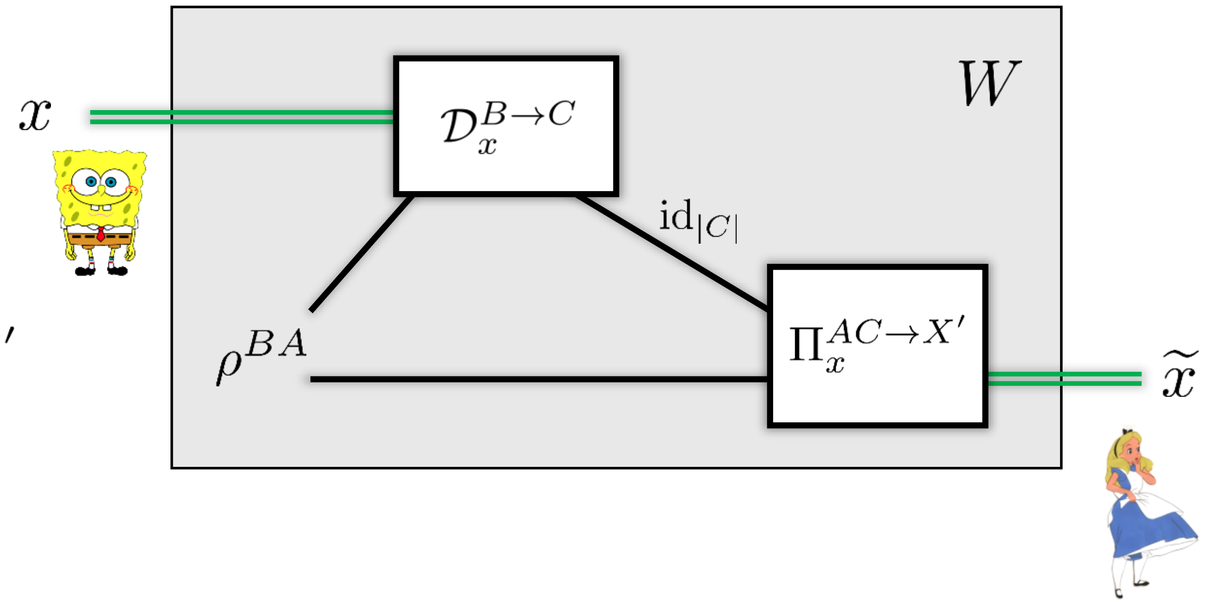

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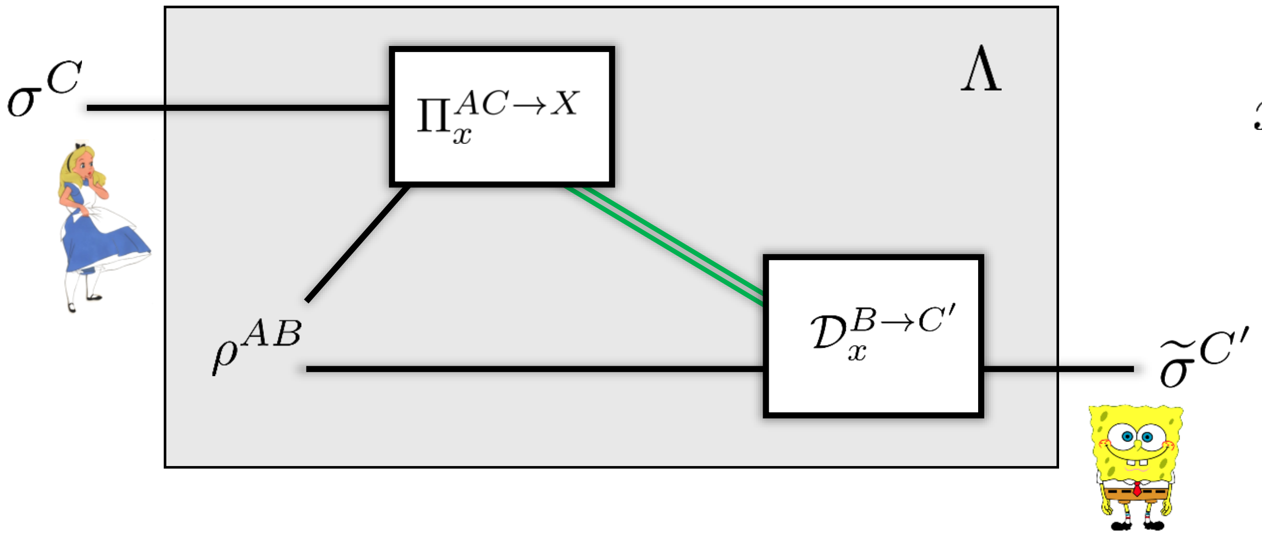

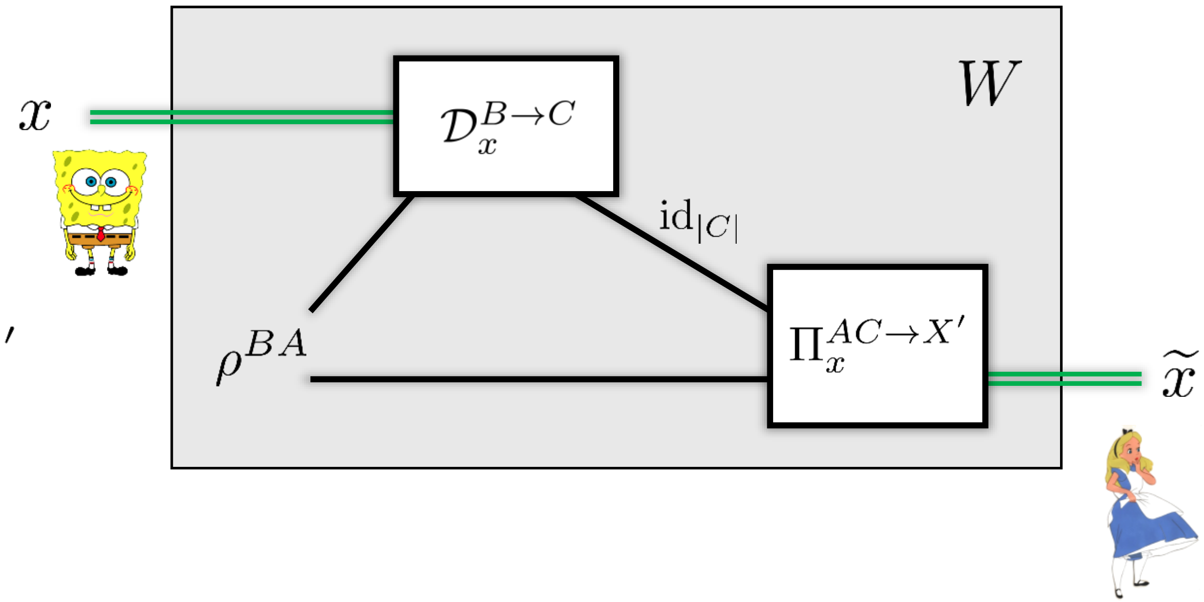

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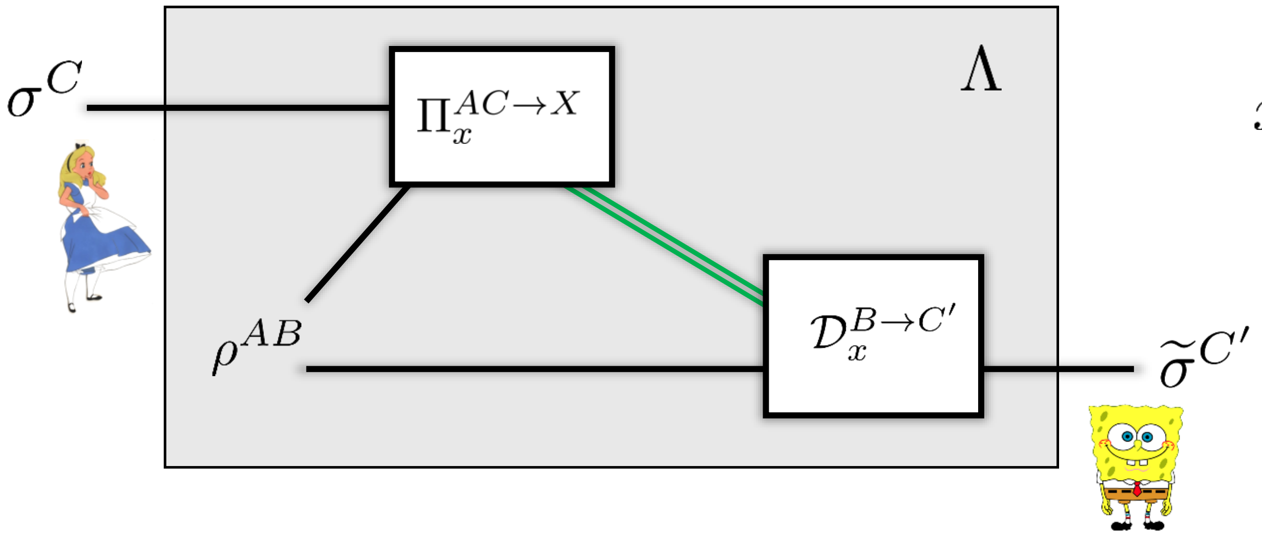

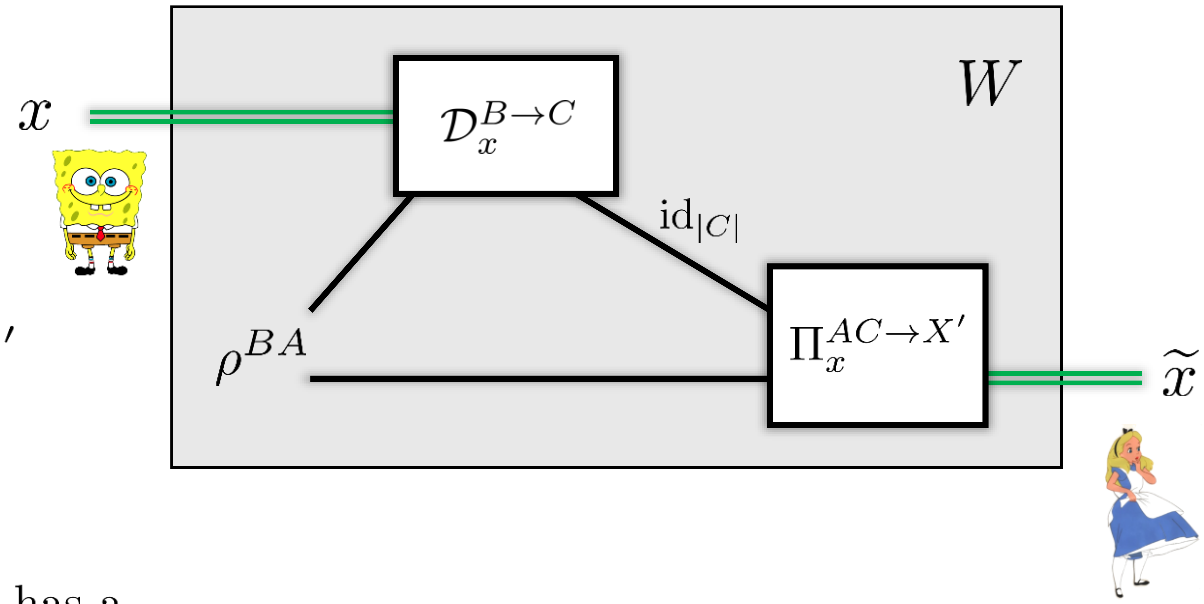

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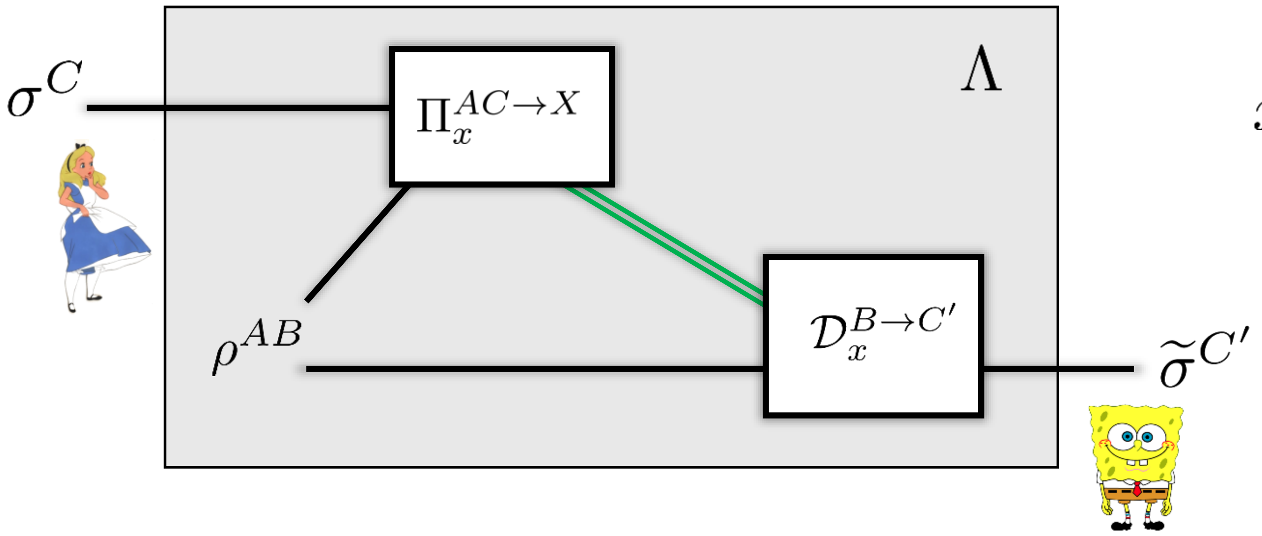

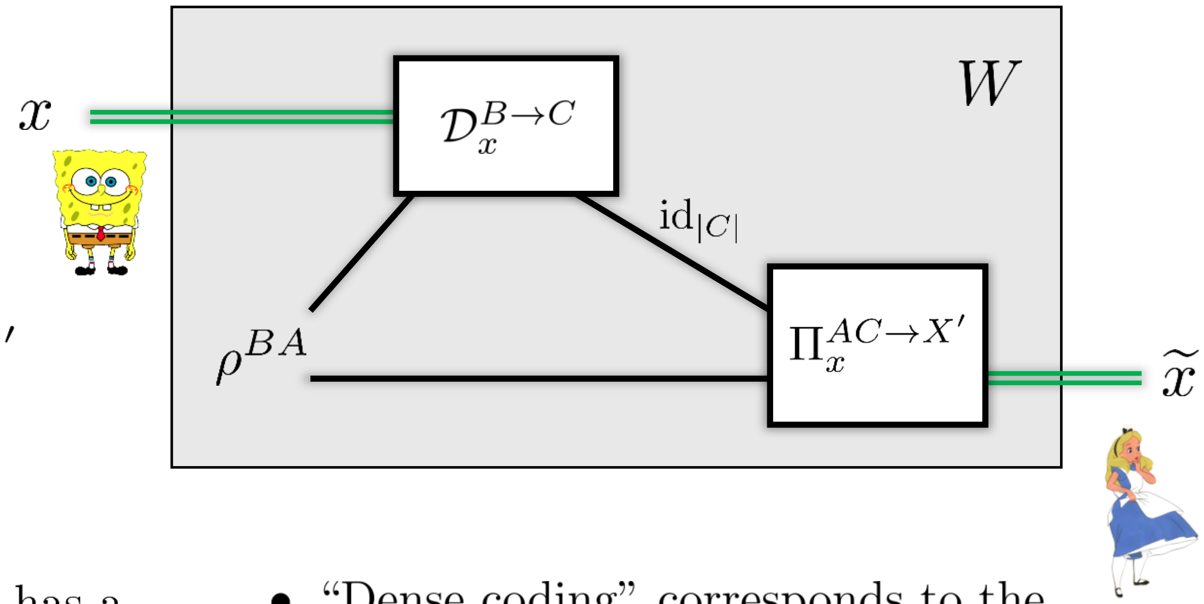

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- "Dense coding" corresponds to the situation when $N > |C|$.

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- Let $(\gamma^+)^{XX'} = \frac{1}{N}\sum_{x=1}^{N} |x\rangle\langle x|^X \otimes |x\rangle\langle x|^{X'}$ define an N-dimensional **maximally correlated** state.

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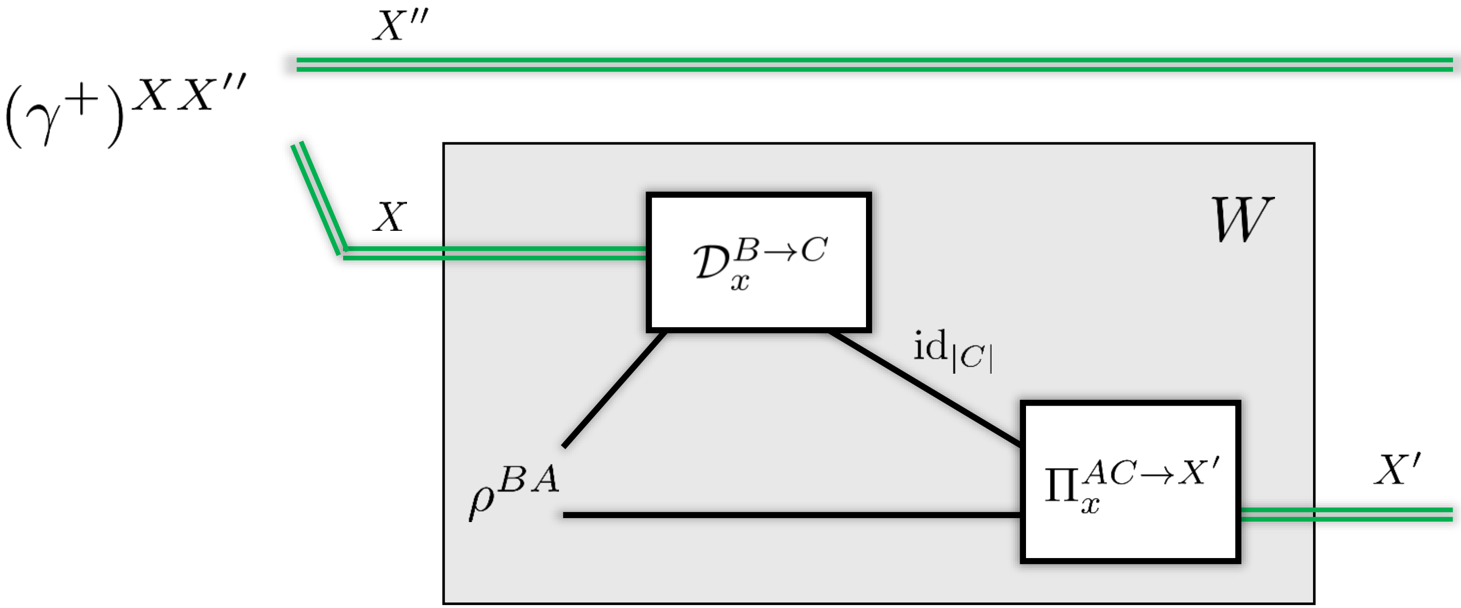

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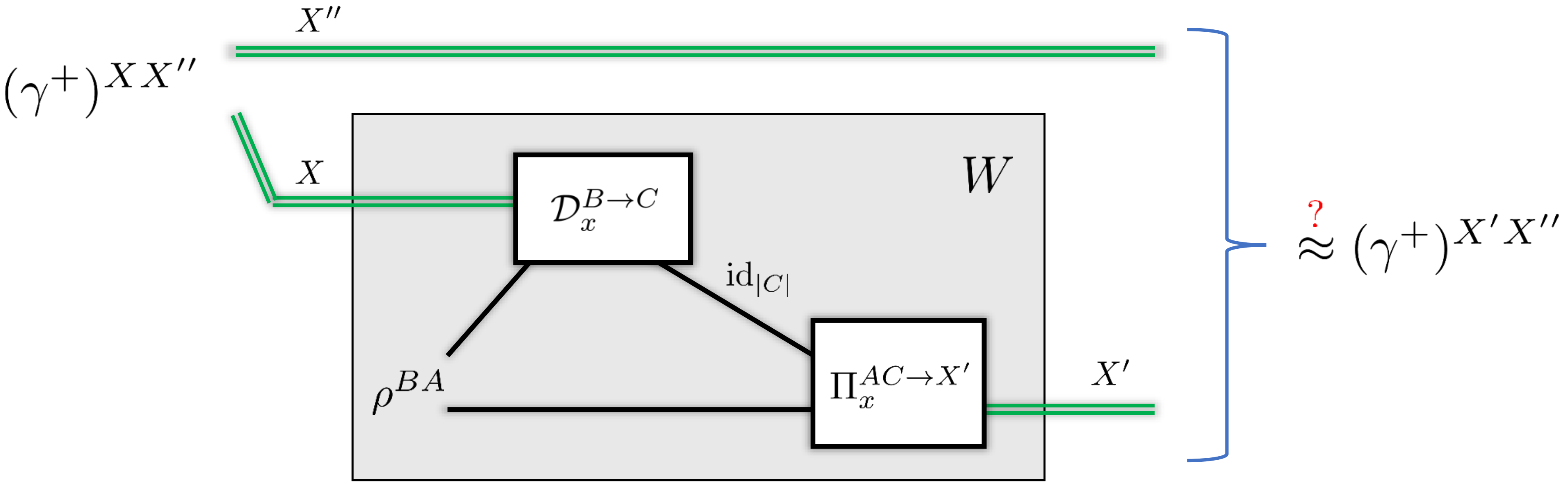

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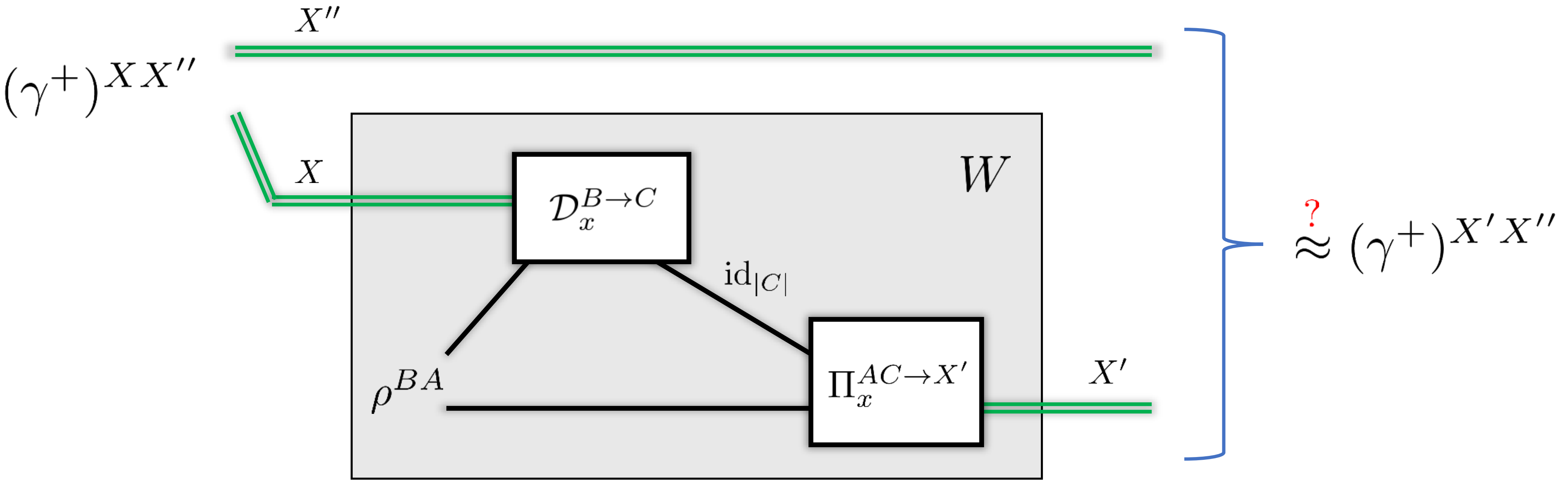

- One way to measure the quality of the classical channel is how well it preserves the maximally correlated (classical) state.

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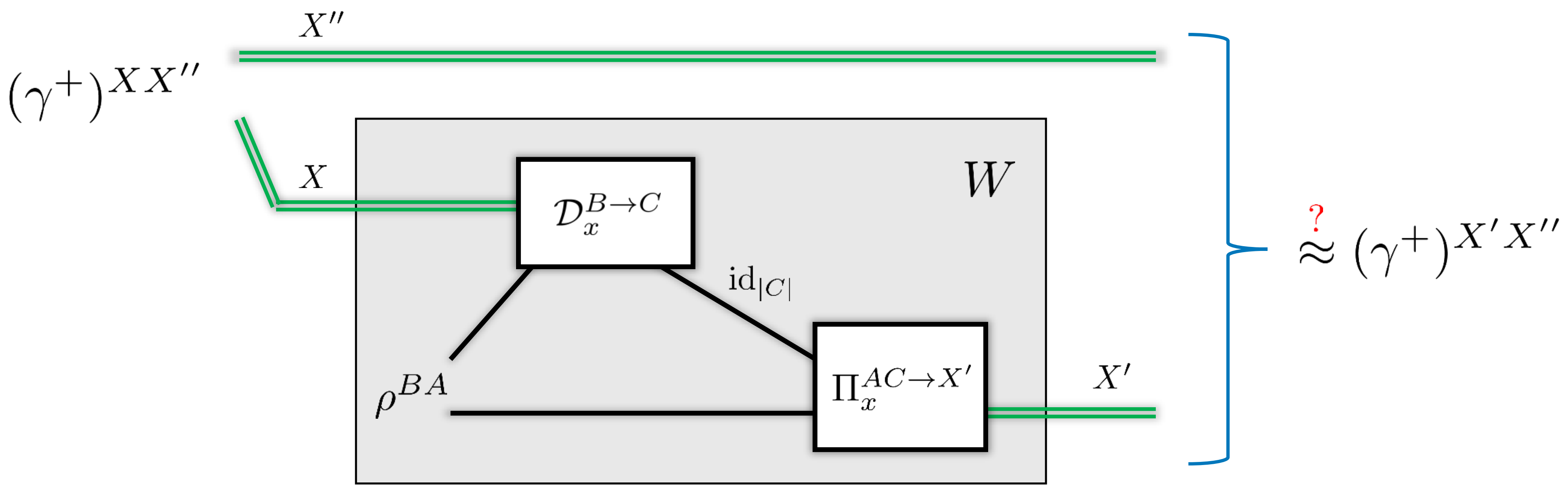

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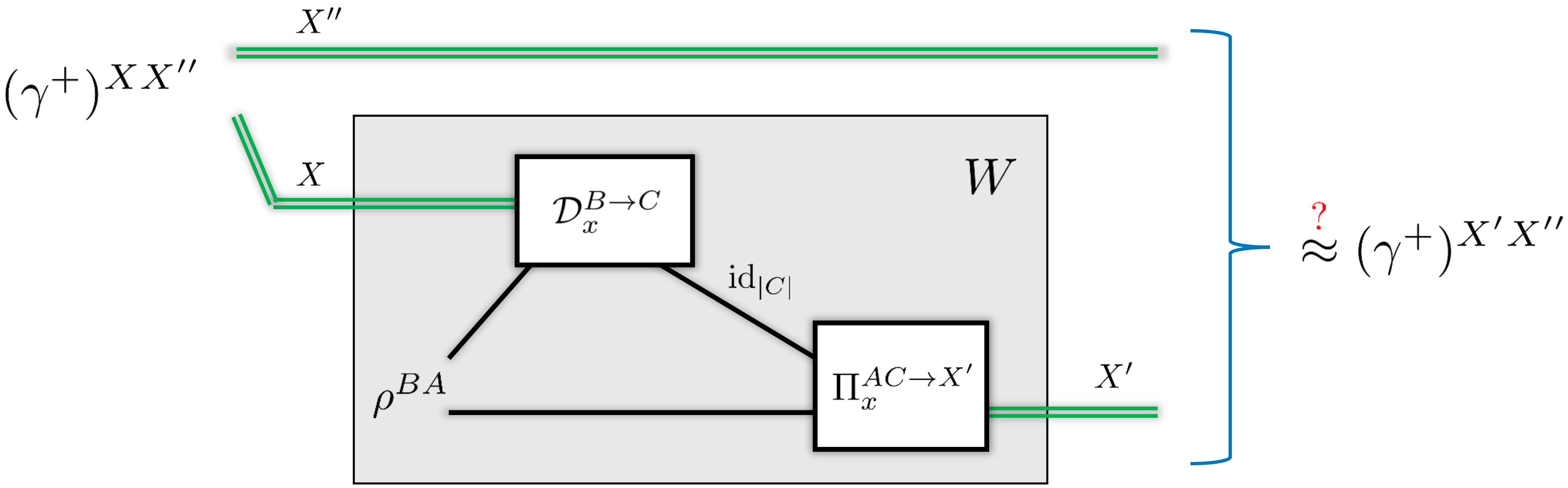

- Analogous to the entanglement fidelity of a quantum channel, we define the **classical correlation fidelity** of a channel $W: X \to X'$ as

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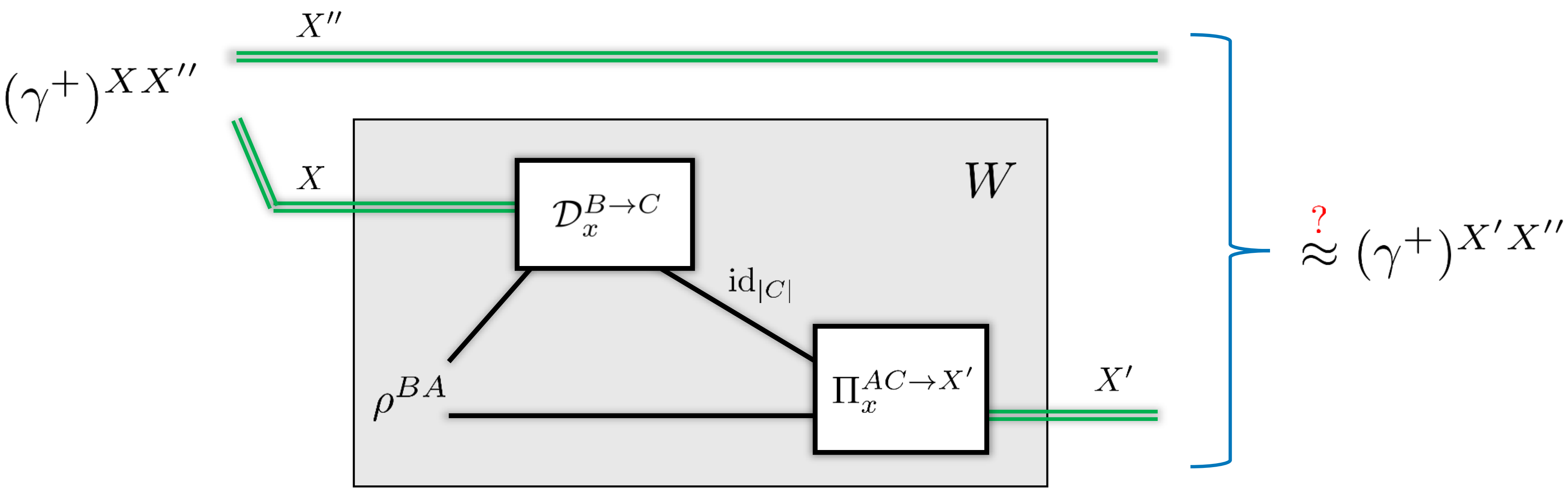

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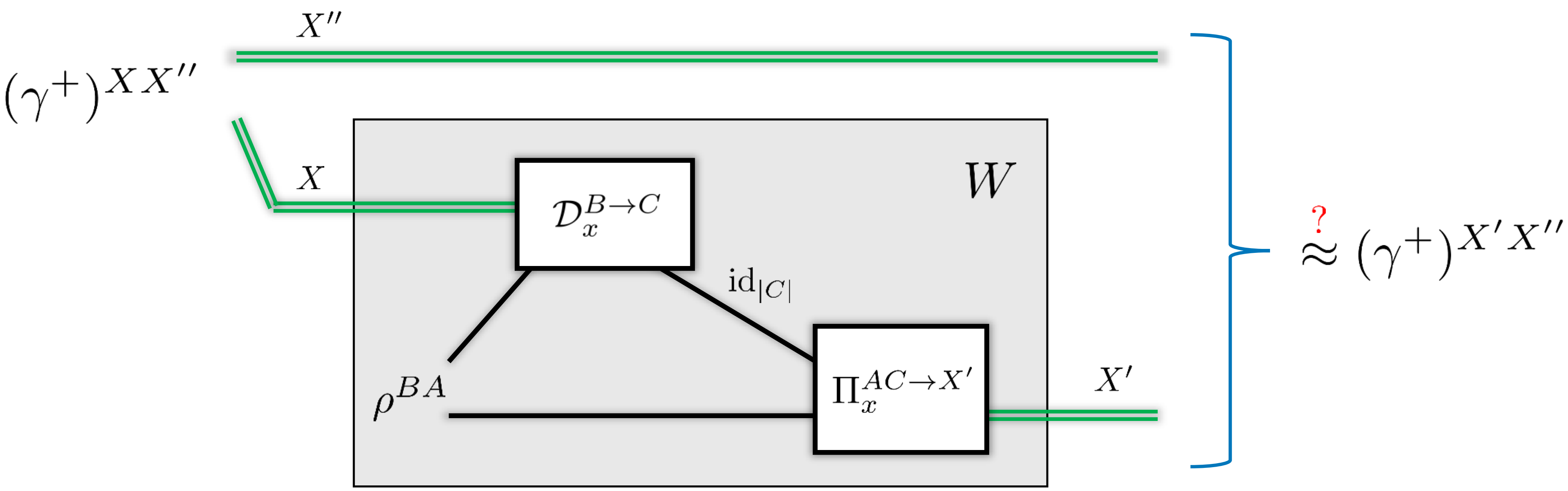

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Interpretation: The classical correlation fidelity of channel W is equivalent to how well the channel transmits a randomly chosen message $x \in \{1, \cdots, N\}$.

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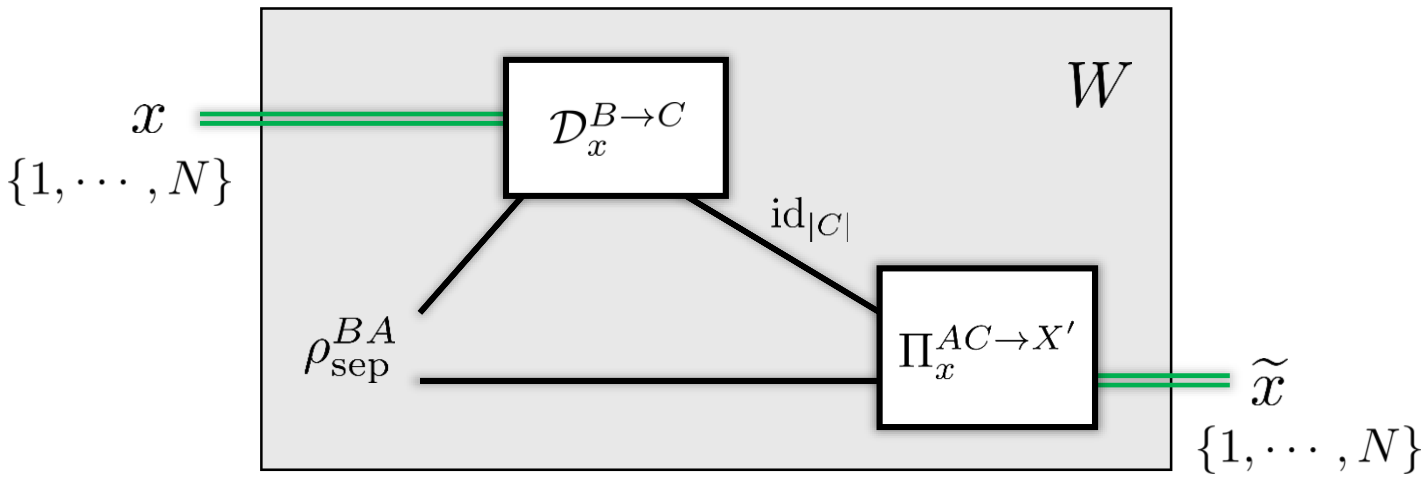

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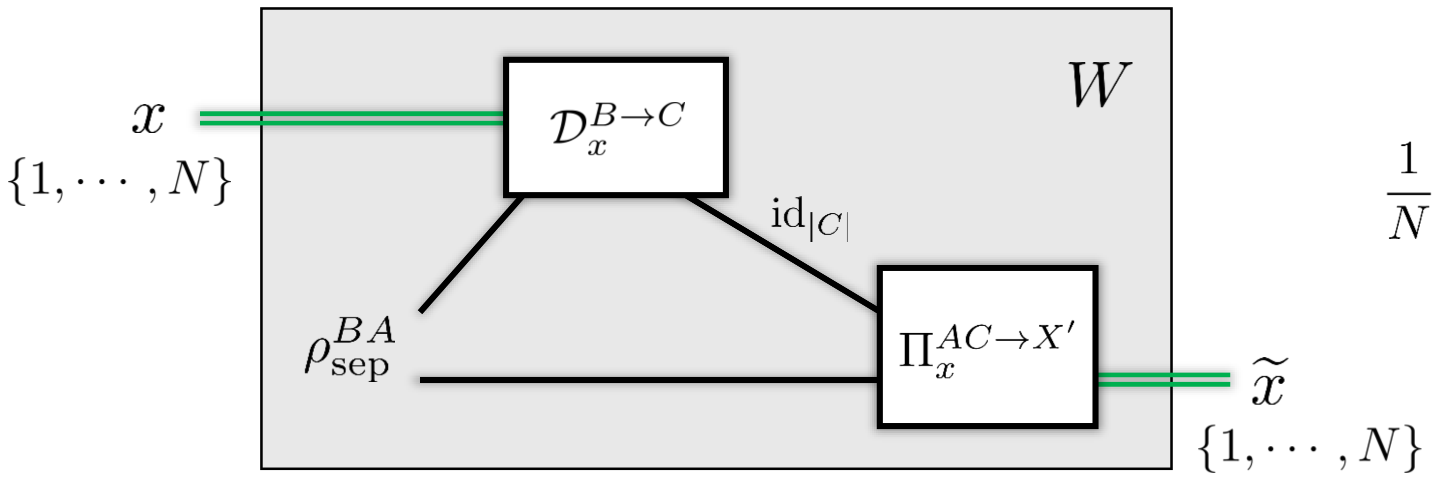

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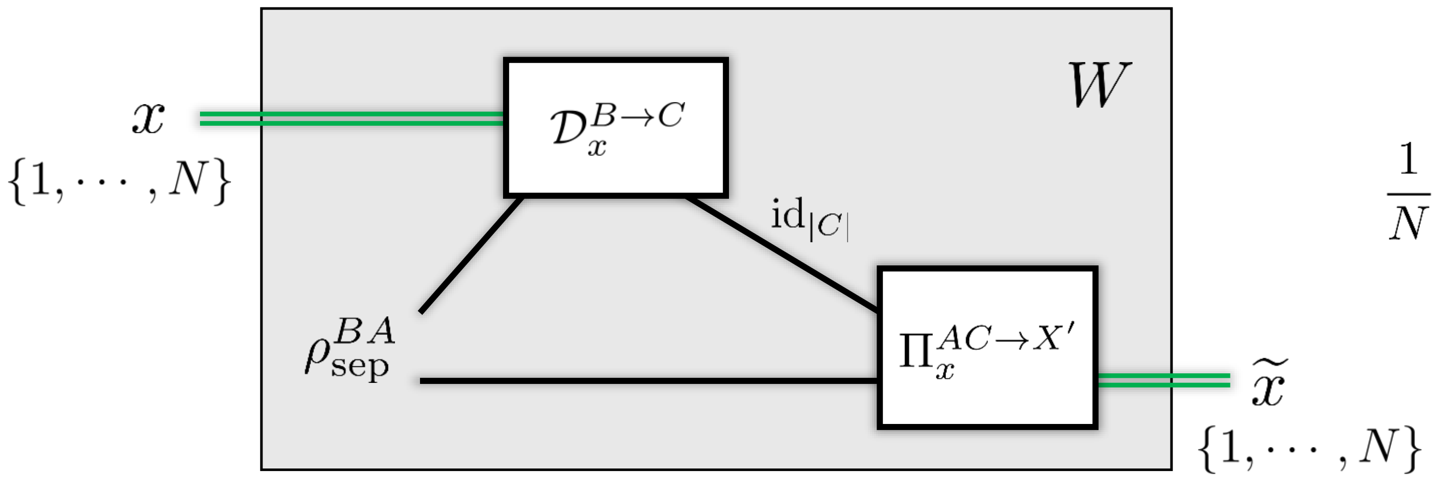

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Thank you for your attention!

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