Strong converse theorems using Rényi entropies

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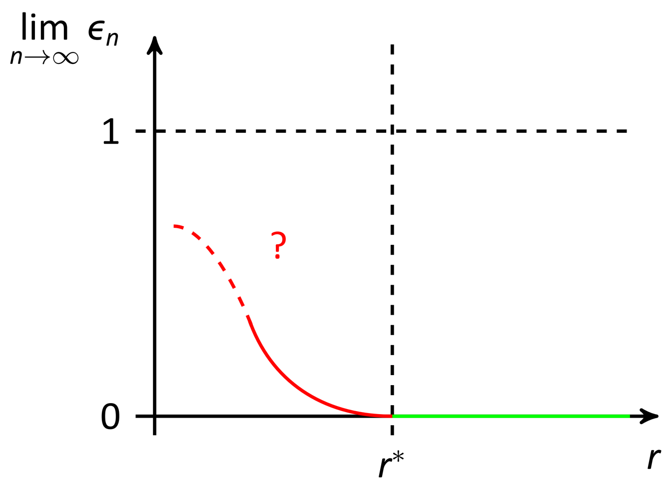
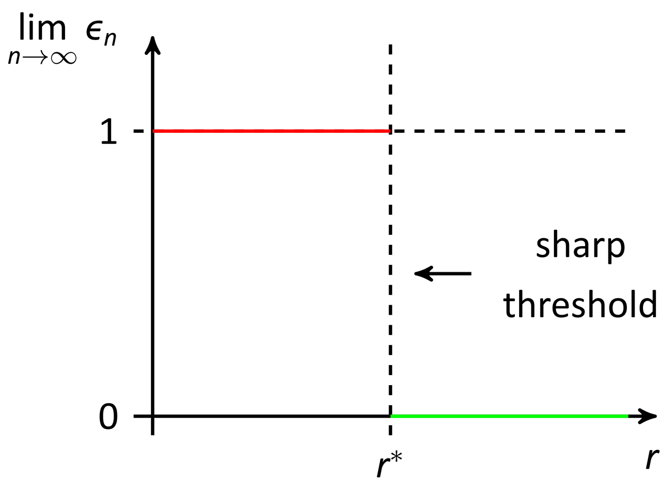
Weak vs. strong converse

Consider an information-theoretic task that takes n copies of some input state ρ, and accomplishes the goal of the task up to an error ϵ_n consuming an available resource at a rate r. If there is a protocol (or code) satisfying $\lim_{n\to\infty}\epsilon_n = 0$, we say that r is an achievable rate. The **optimal rate** r^* is the infimum over all achievable rates. A **coding theorem** establishes $r^* = f(\rho)$ for some entropic quantity f.



Weak converse:
If $r < r^*$ then $\lim_{n\to\infty}\epsilon_n > 0$.



Strong converse:
If $r < r^*$ then $\lim_{n\to\infty}\epsilon_n = 1$.

How can we prove strong converse theorems?
Find bounds on the error ϵ_n in terms of Rényi entropies.

Rényi entropies

- **Definition:** For $\alpha \in (0,\infty)\setminus\{1\}$ and states ρ,σ with $\operatorname{supp}\rho \subseteq \operatorname{supp}\sigma$, the **sandwiched Rényi divergence of order α** [MDS+13, WWY14] is defined as
$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log\operatorname{Tr}\left(\sigma^{(1-\alpha)/(2\alpha)}\rho\,\sigma^{(1-\alpha)/(2\alpha)}\right)^\alpha .$$
- **Data processing inequality:** For a quantum channel Λ and $\alpha \geq 1/2$, we have [Bei13, FL13]
$$\widetilde{D}_\alpha(\rho\|\sigma) \geq \widetilde{D}_\alpha(\Lambda(\rho)\|\Lambda(\sigma)).$$
- **Limit property:** $\lim_{\alpha\to1}\widetilde{D}_\alpha(\rho\|\sigma) = D(\rho\|\sigma) = \operatorname{Tr}[\rho(\log\rho - \log\sigma)]$.
- **Derived quantities:**
  - ▷ Rényi entropy: $S_\alpha(A)_\rho = -\widetilde{D}_\alpha(\rho_A\|I_A)$
  - ▷ Rényi conditional entropy (RCE): $S_\alpha(A|B)_\rho := -\min_{\sigma_B}\widetilde{D}_\alpha(\rho_{AB}\|I_A\otimes\sigma_B)$
  - ▷ Rényi mutual information (RMI): $I_\alpha(A;B)_\rho := \min_{\sigma_B}\widetilde{D}_\alpha(\rho_{AB}\|\rho_A\otimes\sigma_B)$
- RCE and RMI inherit the data processing inequality from $\widetilde{D}_\alpha(\cdot\|\cdot)$, and we have
$$\lim_{\alpha\to1} S_\alpha(A)_\rho = S(\rho) = -\operatorname{Tr}(\rho_A\log\rho_A)$$
$$\lim_{\alpha\to1} S_\alpha(A|B)_\rho = S(A|B)_\rho = S(AB)_\rho - S(B)_\rho$$
$$\lim_{\alpha\to1} I_\alpha(A;B)_\rho = I(A;B)_\rho = S(A)_\rho - S(A|B)_\rho$$

Extending the Rényi entropy calculus

Dimension bounds

For $\alpha \geq 1/2$ and a tripartite state ρ_{ABC} with C quantum,
$$S_\alpha(A|BC)_\rho + 2\log|C| \geq S_\alpha(A|B)_\rho \qquad I_\alpha(A;B)_\rho + 2\log|C| \geq I_\alpha(A;BC)_\rho.$$
whereas for ρ_{ABX} with X classical,
$$S_\alpha(A|BX)_\rho + \log|X| \geq S_\alpha(A|B)_\rho \qquad I_\alpha(A;B)_\rho + \log|X| \geq I_\alpha(A;BX)_\rho.$$

Uncorrelated states

For $\alpha \geq 1/2$ and product states $\tau_{AB}\otimes\theta_C$,
$$S_\alpha(A|BC)_{\tau\otimes\theta} = S_\alpha(A|B)_\tau \qquad I_\alpha(A;BC)_{\tau\otimes\theta} = I_\alpha(A;B)_\tau.$$

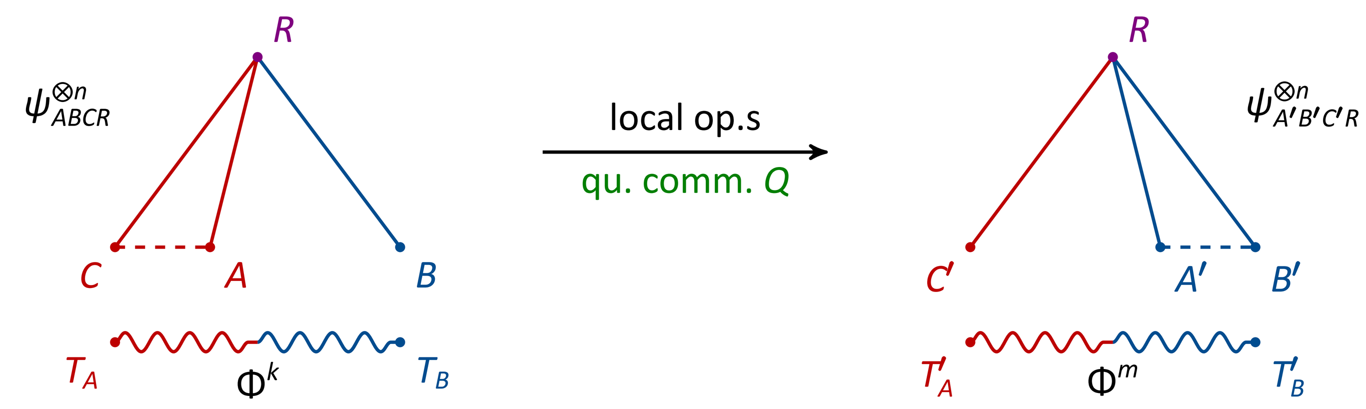
Fidelity bounds

For $\alpha \in (1/2,1)$, $\beta = \alpha/(2\alpha-1)$, and bipartite states ρ_{AB} and σ_{AB},
$$S_\alpha(A|B)_\rho - S_\beta(A|B)_\sigma \geq \frac{2\alpha}{1-\alpha}\log F(\rho_{AB},\sigma_{AB}) \qquad \text{where } F(\omega,\tau) := \left\|\sqrt{\omega}\sqrt{\tau}\right\|_1$$
$$I_\beta(A;B)_\rho - I_\alpha(A;B)_\sigma \geq \frac{2\alpha}{1-\alpha}\log F(\rho_{AB},\sigma_{AB}).$$

Discarding classical information

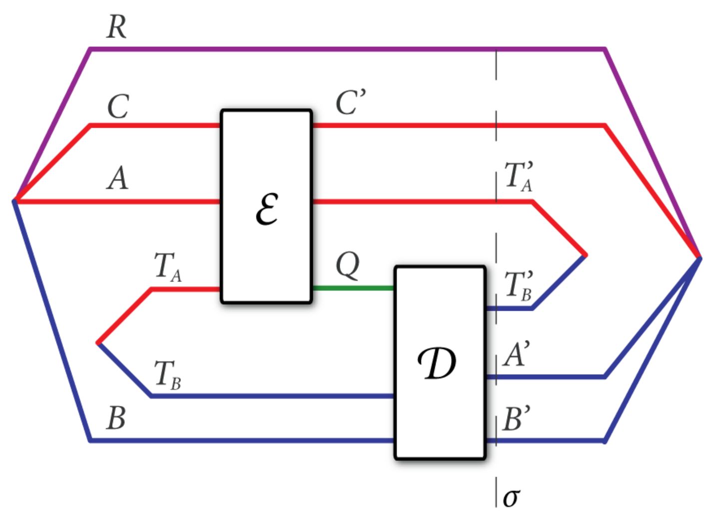
For $\alpha > 0$ and a tripartite state ρ_{ABX} with X classical,
$$S_\alpha(AX|B)_\rho \geq S_\alpha(A|B)_\rho.$$

State redistribution



Initial state: $\psi^{\otimes n}\otimes\Phi^k$ **Target state:** $\psi^{\otimes n}\otimes\Phi^m$

- **Initial situation:** Alice (AC, T_A) and Bob (B, T_B) share n copies of a tripartite state ρ_{ABC} (purified by ψ_{ABCR} where R is inaccessible) and a maximally entangled state (MES) $\Phi^k_{T_AT_B}$ of Schmidt rank k.
- **Goal:** Transfer A system to Bob while preserving all correlations with R, using local operations (encoding $\mathcal{E}$ and decoding $\mathcal{D}$) and noiseless quantum communication (Q) from Alice to Bob.
- **Target state:** n copies of $\rho_{A'B'C'}$ (where A' is with Bob) and some MES $\Phi^m_{T'_AT'_B}$ of Schmidt rank m.



State redistribution protocol.

- **Figure of merit:** Fidelity $F_n := F\left(\psi^{\otimes n}\otimes\Phi^m, \sigma_n\right)$, where σ_n is the final state of the protocol.
- **Quantum communication cost:** $q := \frac{1}{n}\log|Q|$
- **Entanglement cost:** $e := \frac{1}{n}(\log|T_A| - \log|T'_A|) = \frac{1}{n}(\log k - \log m)$
- **Optimal rates:** [LD09, YD09] We have $\lim_{n\to\infty}F_n = 1$ if and only if
$$q \geq \frac{1}{2}I(A;R|B)_\rho \qquad q + e \geq S(A|B)_\rho,$$
where $I(A;R|B)_\rho = S(R|B)_\rho - S(R|AB)_\rho$ is the conditional mutual information.

Strong converse theorem

Strong converse bounds on fidelity

Let $\alpha \in (1/2,1)$ and set $\beta = \alpha/(2\alpha-1)$ and $\kappa(\alpha) = (1-\alpha)/2\alpha$. For every state redistribution protocol with initial state ρ_{ABC} as described above, the fidelity F_n satisfies the following bounds:
$$F_n \leq \exp\left\{-n\kappa(\alpha)\left[S_\beta(AB)_\rho - S_\alpha(B)_\rho - (q+e)\right]\right\}$$
$$F_n \leq \exp\left\{-n\kappa(\alpha)\left[S_\beta(R|B)_\rho - S_\alpha(R|AB)_\rho - 2q\right]\right\}$$
As an alternative to the second inequality, we have
$$F_n \leq \exp\left\{-n\kappa(\alpha)\left[I_\alpha(R;AB)_\rho - I_\beta(R;B)_\rho - 2q\right]\right\}.$$

- The quantity $S_\beta(AB)_\rho - S_\alpha(B)_\rho$ is a Rényi generalization of the conditional entropy $S(A|B)_\rho$ in the sense that $S_\beta(AB)_\rho - S_\alpha(B)_\rho \xrightarrow{\alpha\to1} S(A|B)_\rho$.
- Assume that $q+e < S(A|B)_\rho$ (i.e. rate is in **converse region**). Then, there is an $1/2 < \alpha_0 < 1$ such that we have $\kappa(\alpha_0)\left[S_{\beta_0}(AB)_\rho - S_{\alpha_0}(B)_\rho - (q+e)\right] > 0$ where $\beta_0 = \alpha_0/(2\alpha_0-1)$.
- The same reasoning applies to the bounds above involving $2q$, leading to the following theorem:

Strong converse theorem for state redistribution (see also* [BCT14])

Let ρ_{ABC} be a tripartite state. For any state redistribution protocol with rates q and e satisfying $q+e < S(A|B)_\rho$ or $q < \frac{1}{2}I(A;R|B)_\rho$, the fidelity decays exponentially fast, i.e. there is a $\delta > 0$ such that for every $n \in \mathbb{N}$ we have $F_n \leq \exp(-n\delta)$.

* While the authors in [BCT14] first proved a strong converse theorem only for q, our original contribution was the corresponding theorem for $q+e$. Both papers now include the full theorem as stated above.

- In [LWD15] we also prove similar strong converse theorems for:
  - ▷ State redistribution with feedback (allowing back-communication from Bob to Alice)
  - ▷ Measurement compression with quantum side information (QSI)
  - ▷ Randomness extraction against QSI
  - ▷ Data compression with QSI

References: **[BCT14]** M. Berta, M. Christandl, D. Touchette, arXiv:1409.4338 [quant-ph] (2014). **[Bei13]** S. Beigi, J. Math. Phys. 54.12, p. 122 202 (2013). **[FL13]** R. Frank, E. Lieb, J. Math. Phys. 54.12, p. 122 201 (2013). **[LD09]** Z. Luo, I. Devetak, IEEE Trans. Inf. Th. 55.3, pp. 1331-1342 (2009). **[LWD15]** F. Leditzky, M.M. Wilde, N. Datta, arXiv:1506.02635 [quant-ph] (2015). **[MDS+13]** M. Müller-Lennert et al., J. Math. Phys. 54.12, p. 122 203 (2013). **[WWY14]** M.M. Wilde, A. Winter, D. Yang, Comm. Math. Phys. 331.2, pp. 593-622 (2014). **[YD09]** J. Yard, I. Devetak, IEEE Trans. Inf. Th. 55.11, pp. 5339-5351 (2009).

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