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The platypus of the quantum channel zoo

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Joint work with D. Leung, V. Siddhu, G. Smith, J. Smolin

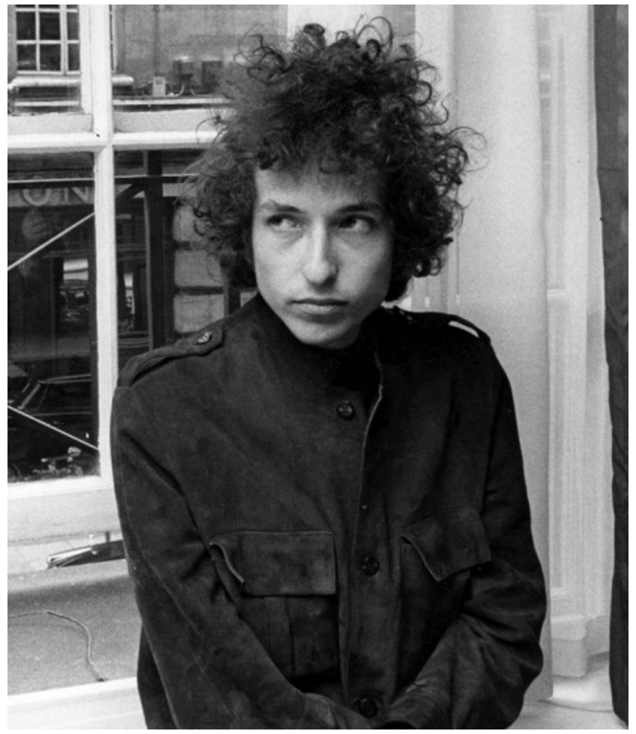
Quantum channels

A quantum channel is a model for a noisy communication link between a quantum sender Alice and a quantum receiver Bob.

Alice

quantum channel

$\mathcal{N}: A \longrightarrow B$

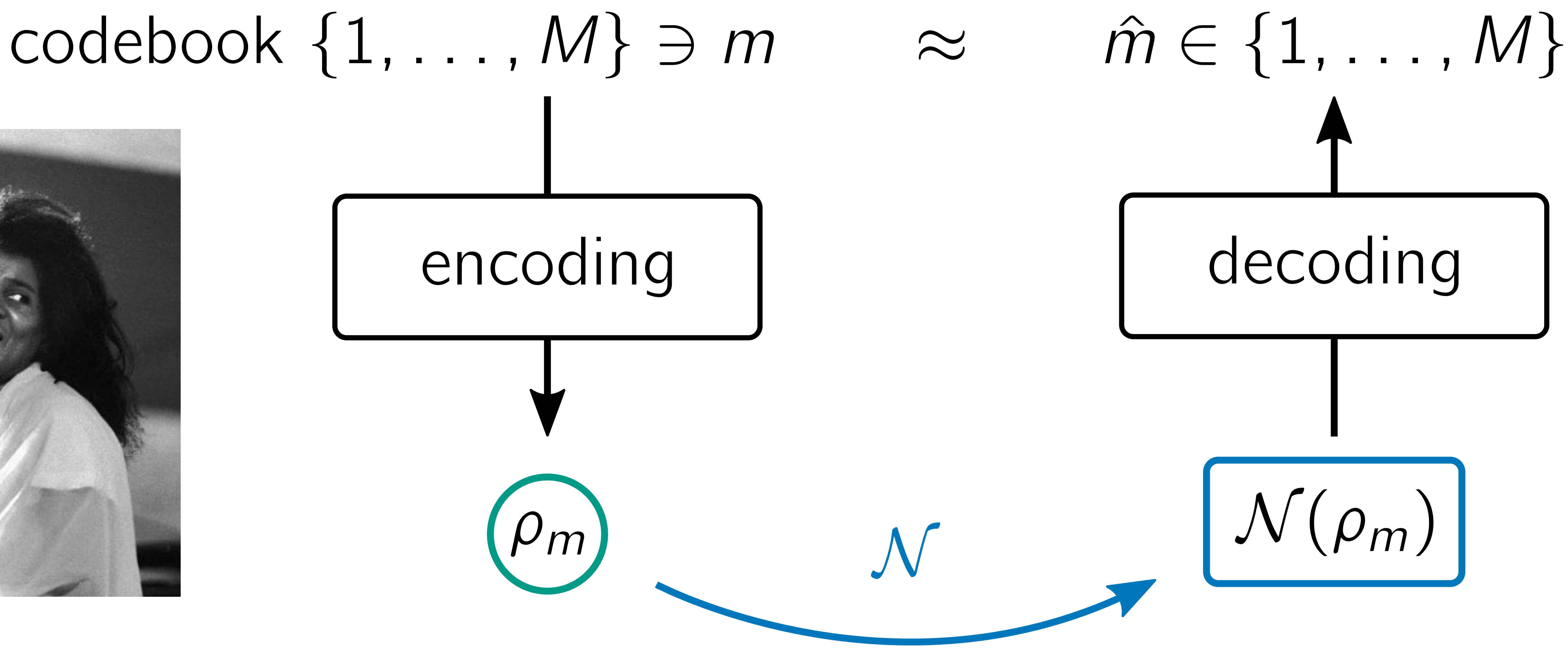


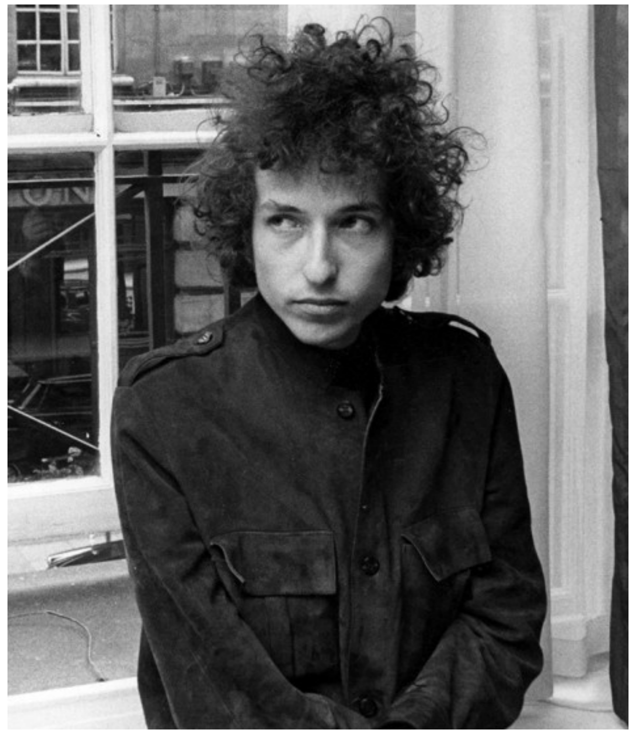
Bob

A quantum channel can transmit different types of information: quantum, private, classical information

Classical information transmission

Alice



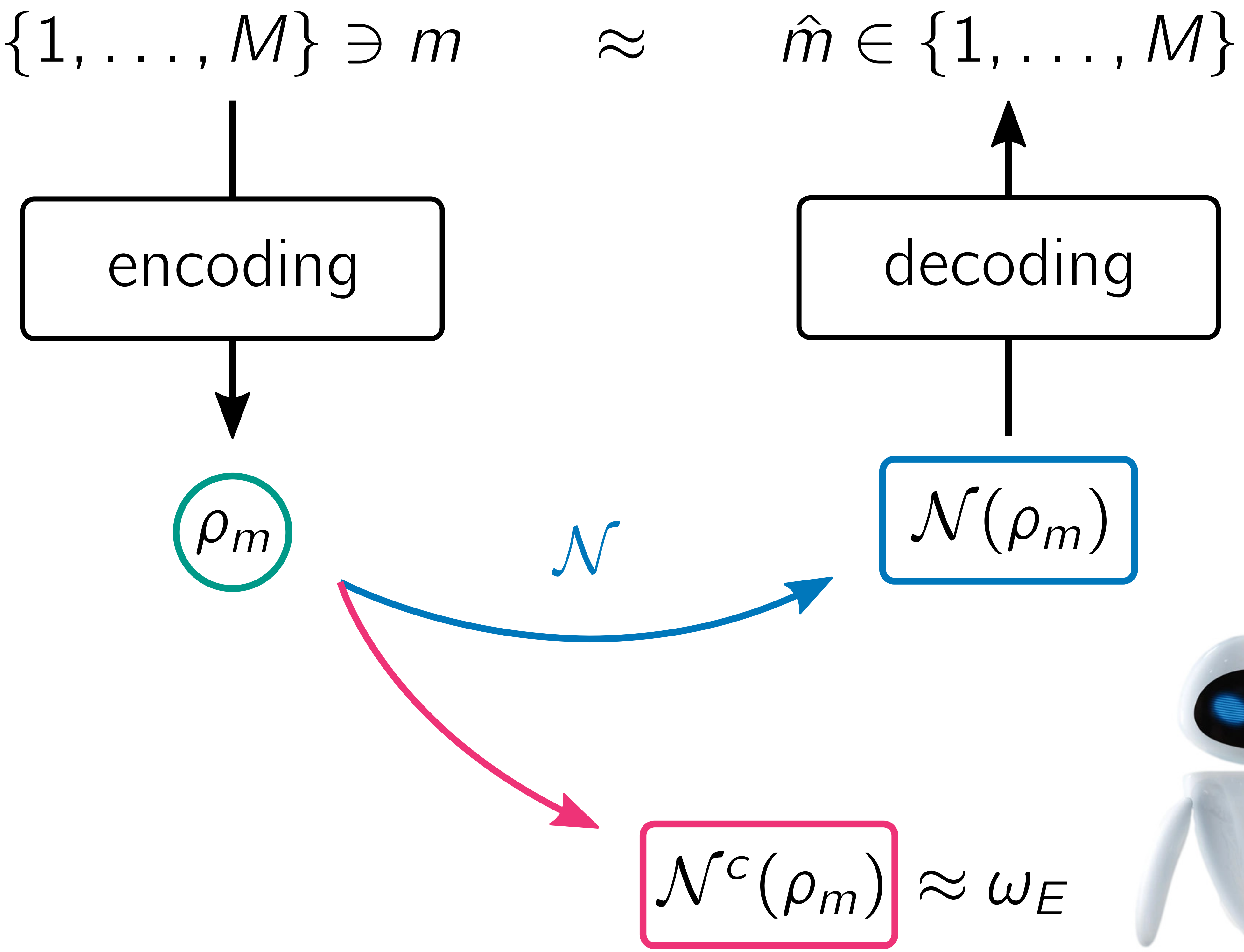


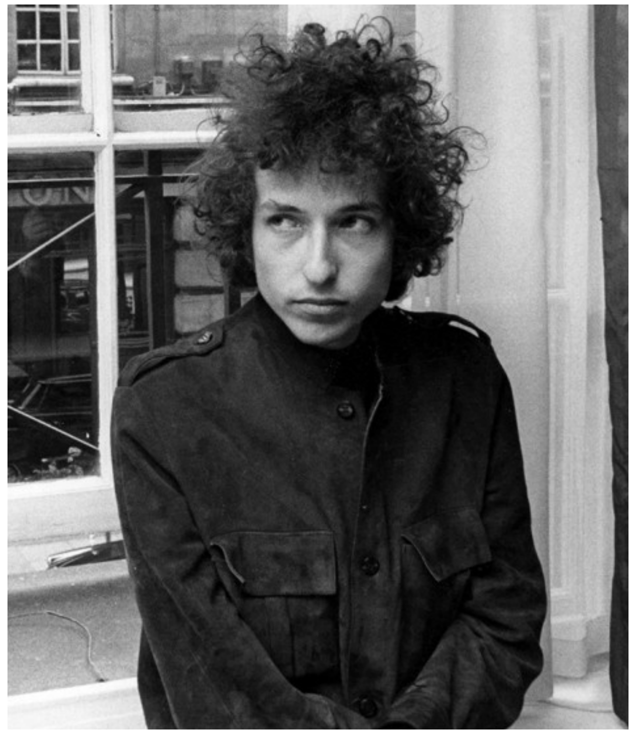
Bob

Relevant quantity: codebook size $\log M$

Private information transmission

Alice





Bob

Eve

Relevant quantity:

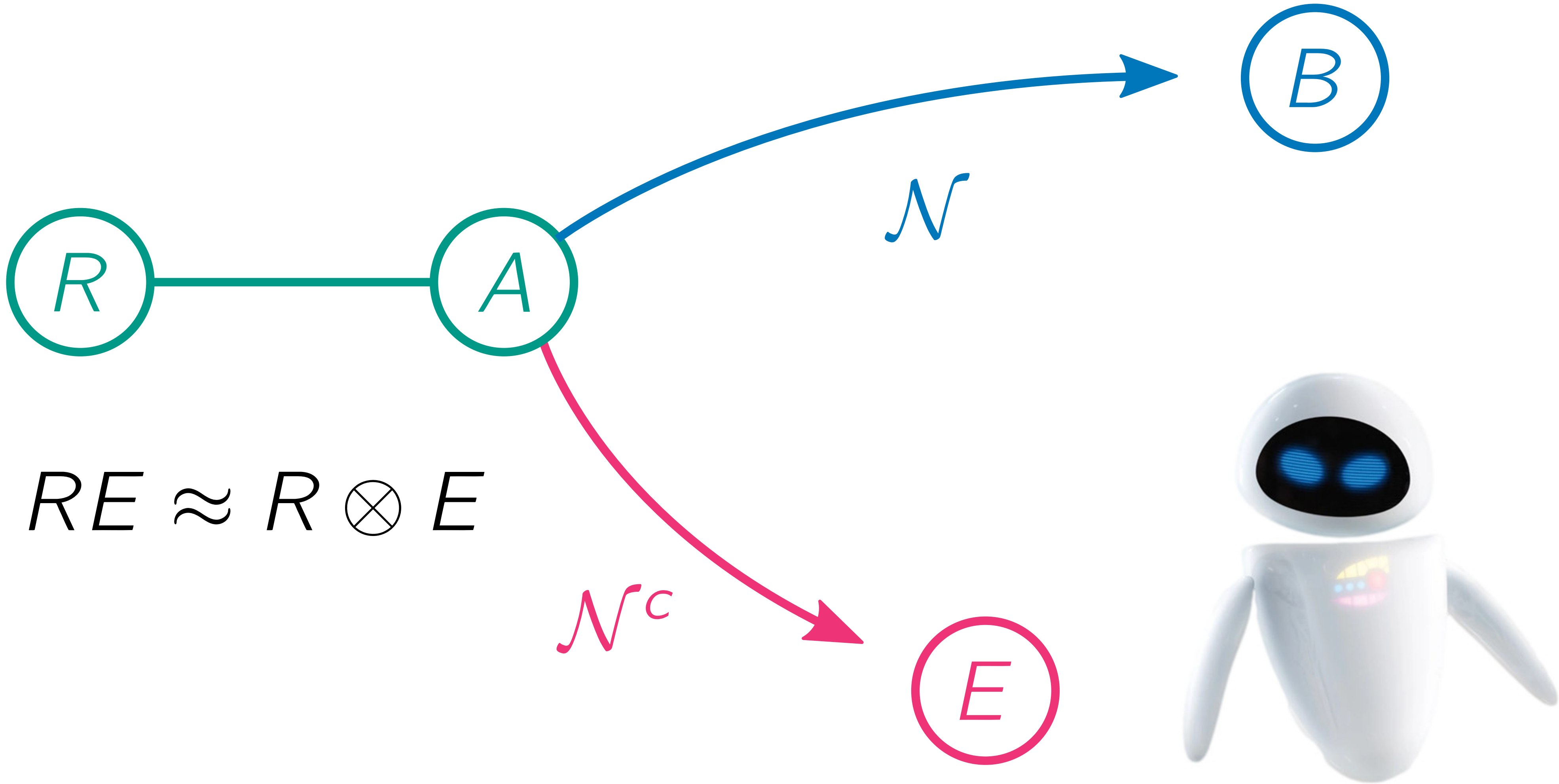
codebook size $\log M$

Quantum information transmission

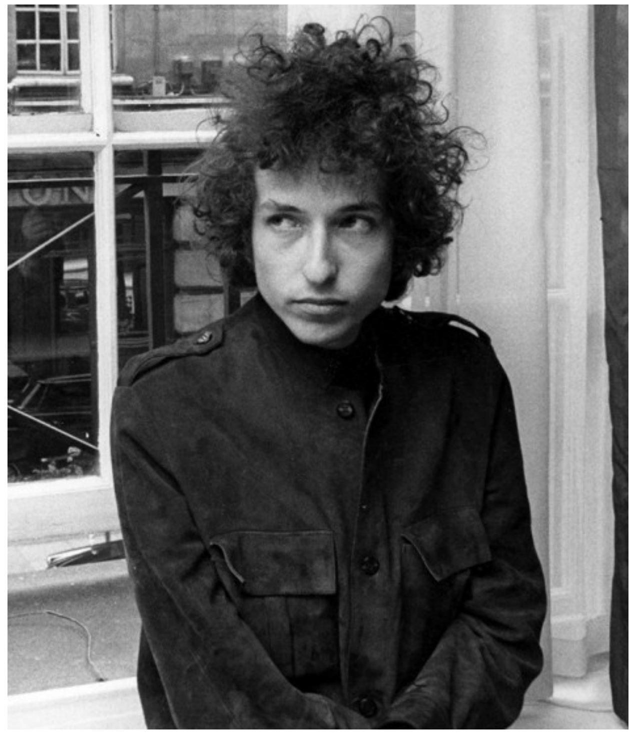
Relevant quantity: subspace size $\log |R|$

$$RB \approx \Phi^+_{RB}$$

Alice



$$RE \approx R \otimes E$$



Bob

Eve

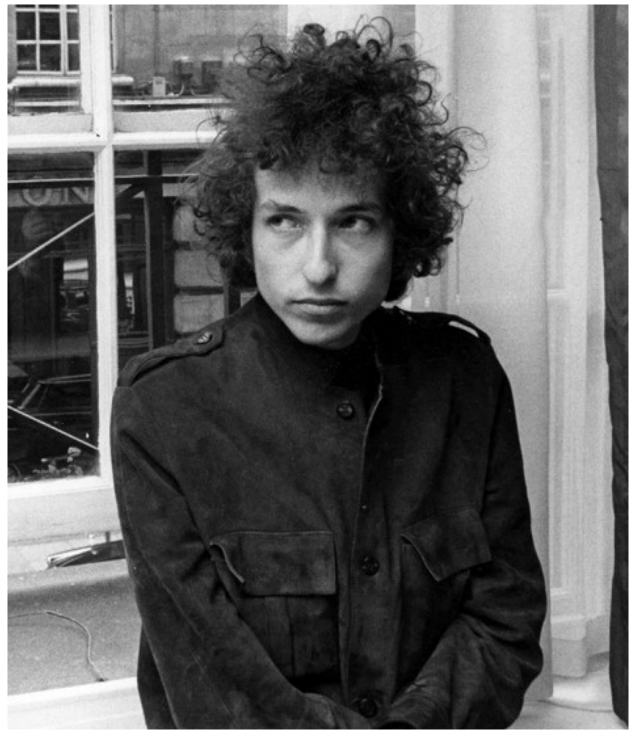
Quantum channel capacities

Channel capacities quantify how much information a channel can transmit faithfully. Curiously, this question may be very hard to answer for quantum channels!

Alice

quantum channel

$\mathcal{N} : A \longrightarrow B$



Bob

Talk outline

1. **Quantum channel capacities and (super-)additivity**

2. The platypus channel and its capacities

3. Strong superadditivity of the platypus channel

4. Further results and open problems

Quantum channel capacities

Quantum channel $\mathcal{N}: A \to B$:

completely positive, trace-preserving linear map $\mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$.

Quantum channel capacity is defined as the optimal rate of

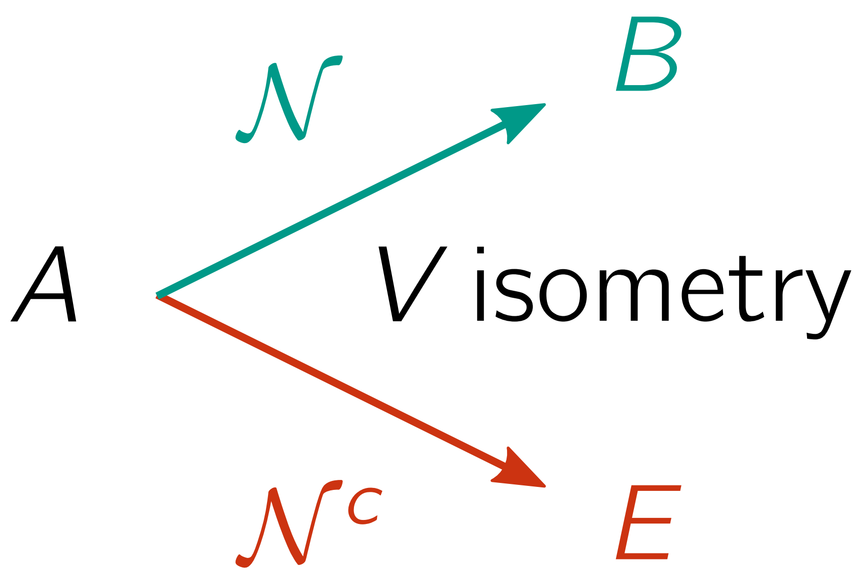
faithful transmission of {quantum, private, classical} information via $\mathcal{N}$.

Quantum capacity		Private capacity		Classical capacity
$Q(\mathcal{N})$	$\leq$	$P(\mathcal{N})$	$\leq$	$C(\mathcal{N})$

Some technical definitions

Complementary channel $\mathcal{N}^c$ models loss to the environment:

If $\mathcal{N}(X) = \mathrm{tr}_E VXV^\dagger$, then $\mathcal{N}^c(X) = \mathrm{tr}_B VXV^\dagger$.



Quantum state: $\rho \in \mathcal{L}(\mathcal{H})$, $\rho \geq 0$, $\mathrm{tr}\,\rho = 1$

Von Neumann entropy: $S(\rho) = -\mathrm{tr}\,\rho \log \rho$

Mutual information: $I(A; B) = S(A) + S(B) - S(AB)$

for a bipartite state ρ_{AB} and $S(X) \equiv S(\rho_X)$

Quantum state ensemble $\{p_i, \rho_A^i\}$ can be encoded in a

classical-quantum state $\rho_{XA} = \sum_i p_i |i\rangle\langle i|_X \otimes \rho_A^i$

with mutual information: $I(X; A) = S(\sum_i p_i \rho_A^i) - \sum_i p_i S(\rho_A^i)$.

Coding theorems for quantum channel capacities

Quantum capacity

$$Q(\mathcal{N}) = \sup_{n\in\mathbb{N}} \frac{1}{n} Q^{(1)}(\mathcal{N}^{\otimes n})$$

with the **coherent information**

$$Q^{(1)}(\mathcal{N}) = \max_{\{p_i,|\psi_i\rangle\}} \{I(X;B) - I(X;E)\}.$$

Private capacity

$$P(\mathcal{N}) = \sup_{n\in\mathbb{N}} \frac{1}{n} P^{(1)}(\mathcal{N}^{\otimes n})$$

with the **private information**

$$P^{(1)}(\mathcal{N}) = \max_{\{p_i,\rho_i\}} \{I(X;B) - I(X;E)\}.$$

Classical capacity

$$C(\mathcal{N}) = \sup_{n\in\mathbb{N}} \frac{1}{n} C^{(1)}(\mathcal{N}^{\otimes n})$$

with the **Holevo information**

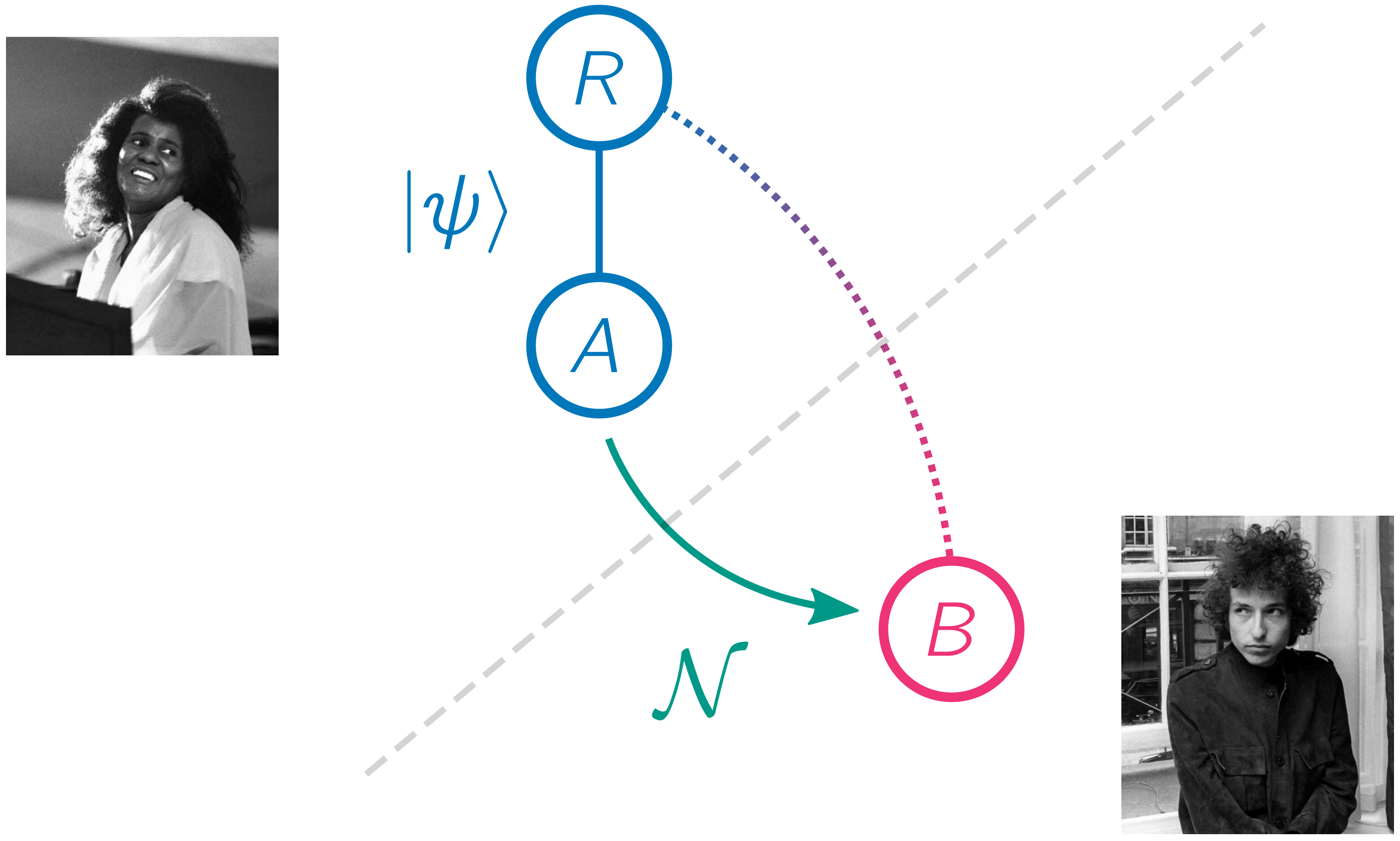
$$C^{(1)}(\mathcal{N}) = \max_{\{p_i,\rho_i\}} I(X;B).$$

[Schumacher & Westmoreland '97, Lloyd '97, Holevo '98, Shor '02, Cai et al. '04, Devetak '05, ,

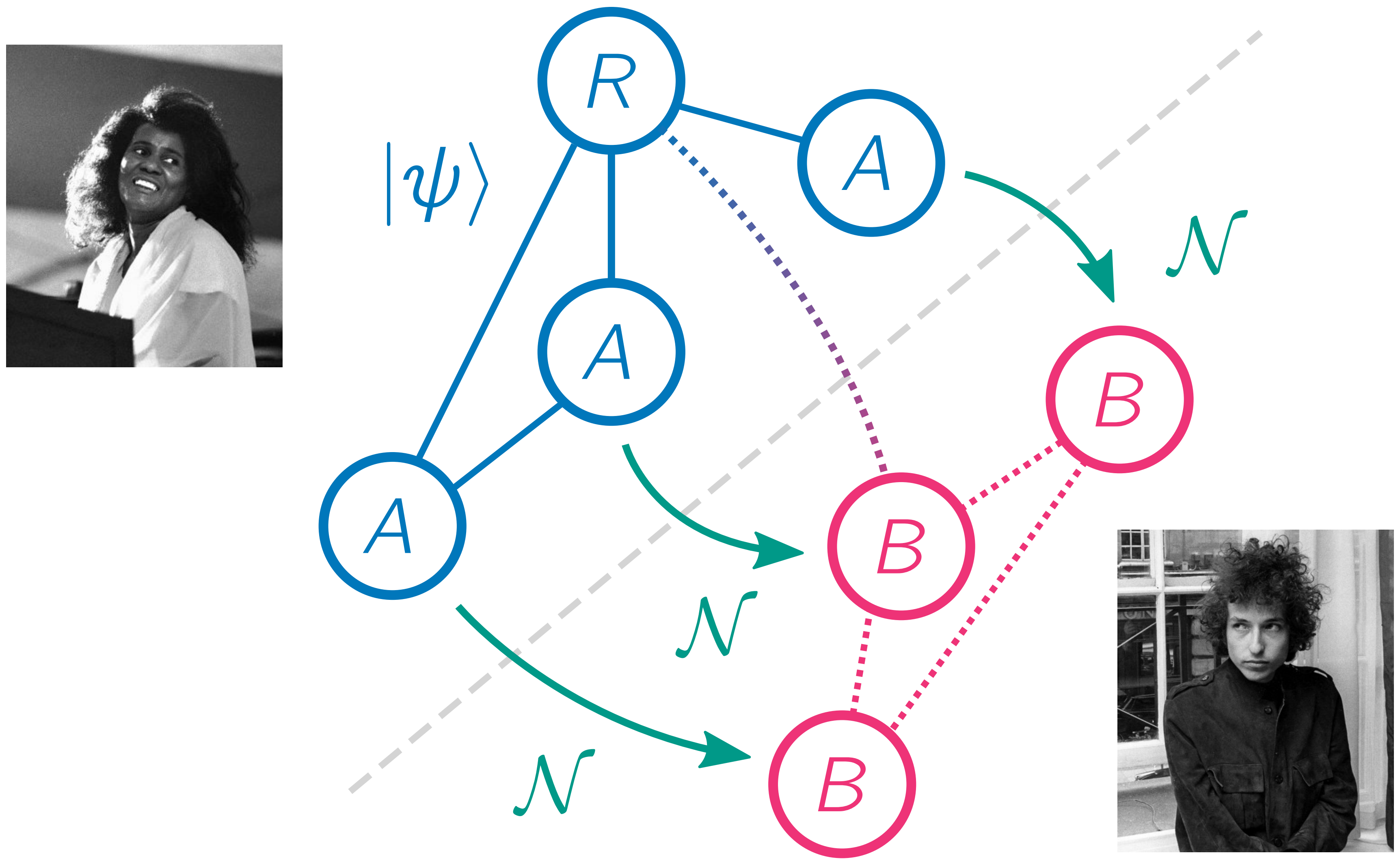
Operational interpretation of coding theorems

Quantum capacity: $Q(\mathcal{N}) = \sup_{n\in\mathbb{N}} \frac{1}{n} Q^{(1)}(\mathcal{N}^{\otimes n})$

Rewrite the coherent information as $Q^{(1)}(\mathcal{N}) = \max_{|\psi\rangle_{RA}} \{S(\mathcal{N}(\psi_A)) - S(\mathrm{id}_R \otimes \mathcal{N}(\psi_{RA}))\}$.



$Q(\mathcal{N}) \geq Q^{(1)}(\mathcal{N}) \geq S(B) - S(RB)$



$Q(\mathcal{N}) \geq \frac{1}{3} Q^{(1)}(\mathcal{N}^{\otimes 3}) \geq \frac{1}{3}(S(B^3) - S(RB^3))$

Superadditivity of information quantities

Superadditivity effects make the regularization in these formulas necessary:

For $F \in \{Q, P, C\}$, we have $F(\mathcal{N}) = \sup_{n\in\mathbb{N}} \frac{1}{n} F^{(1)}(\mathcal{N}^{\otimes n})$,

and there are $\mathcal{N}$ and $n > m$ such that $\frac{1}{n} F^{(1)}(\mathcal{N}^{\otimes n}) > \frac{1}{m} F^{(1)}(\mathcal{N}^{\otimes m})$.

[DiVincenzo et al. '98, Smith et al. '08, Hastings '08]

For the quantum capacity, we can even have the following:

There are channels $\mathcal{N}$ and $\mathcal{M}$ such that $Q(\mathcal{N} \otimes \mathcal{M}) > Q(\mathcal{N}) + Q(\mathcal{M})$.

[Smith & Yard '08, Brandão et al. '12]

Weakly and strongly additive channels

For certain quantum channels the information quantities $F^{(1)}(\cdot)$ are **additive and regularization is not needed**: $F(\mathcal{N}) = F^{(1)}(\mathcal{N})$.

$(F \in \{Q, P, C\})$

We distinguish between two types of additivity:

Weak additivity:

For all $n \in \mathbb{N}$,

$F^{(1)}(\mathcal{N}^{\otimes n}) = nF^{(1)}(\mathcal{N})$.

Strong additivity:

For all channels $\mathcal{M}$,

$F^{(1)}(\mathcal{N} \otimes \mathcal{M}) = F^{(1)}(\mathcal{N}) + F^{(1)}(\mathcal{M})$.

Additivity and non-additivity

There are many results for additivity or lack thereof. (Ask me for references!)

But the situation is rather different for the capacities Q, P, C:

Classical capacity C

Many classes of **strongly additive** channels are known (entanglement-breaking, depolarizing, unital qubit, ...), but **no explicit example of superadditivity**.

Quantum capacity Q, private capacity P

Only three classes of **weakly additive** channels are known (degradable, anti-deg., PPT), but **plenty of explicit examples of superadditivity**.

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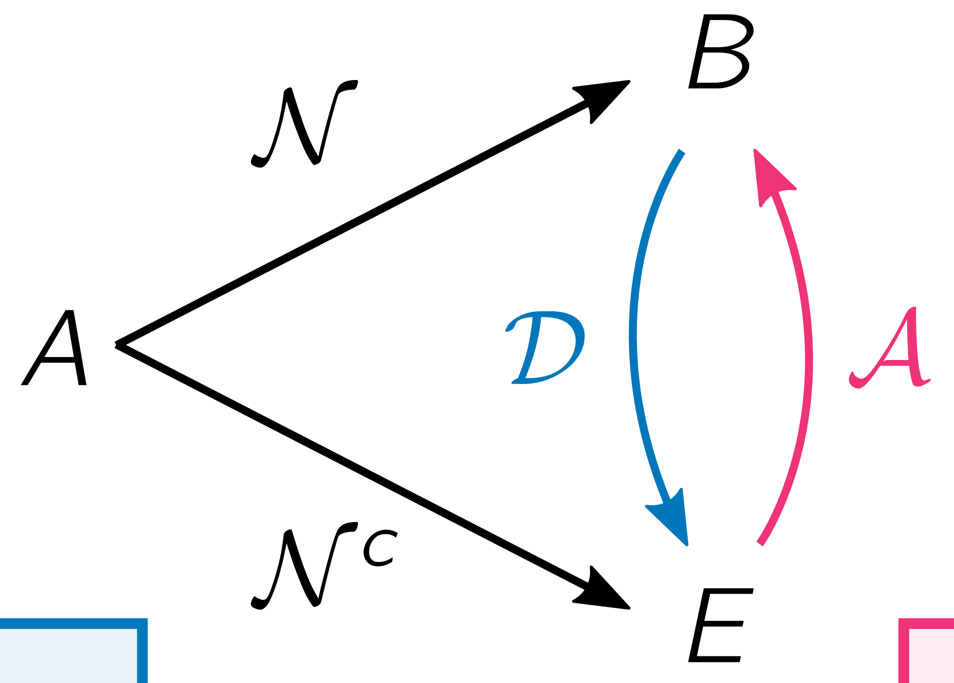
Platypus channel: Stitching simple channels

Consider two simple and additive channels for $s \in [0, 1/2]$:

$$\mathcal{N}_1 : V_1|0\rangle = \sqrt{s}|00\rangle + \sqrt{1-s}|11\rangle$$
$$V_1|1\rangle = |20\rangle$$

$$\mathcal{N}_2 : V_1|1\rangle = |20\rangle$$
$$V_2|2\rangle = |21\rangle$$



$\mathcal{N}_1$ is a **degradable** channel:
There exists a degrading map
$\mathcal{D} : B_1 \to E_1$ s.t. $\mathcal{N}_1^c = \mathcal{D} \circ \mathcal{N}_1$.

$\mathcal{N}_2$ is an **antidegradable** channel:
There exists an antidegrading map
$\mathcal{A} : E_2 \to B_2$ s.t. $\mathcal{N}_2 = \mathcal{A} \circ \mathcal{N}_2^c$.

Platypus channel: Stitching simple channels

$\mathcal{N}_1 : V_1|0\rangle = \sqrt{s}|00\rangle + \sqrt{1-s}|11\rangle$

$V_1|1\rangle = |20\rangle$

$\mathcal{N}_2 : V_1|1\rangle = |20\rangle$

$V_2|2\rangle = |21\rangle$

Capacities of each of the channels are known:

$Q^{(1)}(\mathcal{N}_1) = Q(\mathcal{N}_1) = P(\mathcal{N}_1) = f(s),\ C(\mathcal{N}_1) = 1$

$Q(\mathcal{N}_2) = P(\mathcal{N}_2) = C(\mathcal{N}_2) = 0$

Platypus channel: Stitch $\mathcal{N}_1$ and $\mathcal{N}_2$ together along $|1\rangle$.

$$V|0\rangle = \sqrt{s}|00\rangle + \sqrt{1-s}|11\rangle$$
$$V|1\rangle = |20\rangle$$
$$V|2\rangle = |21\rangle$$

[Wang & Duan '18, Siddhu '21]

Weak additivity of the platypus channel

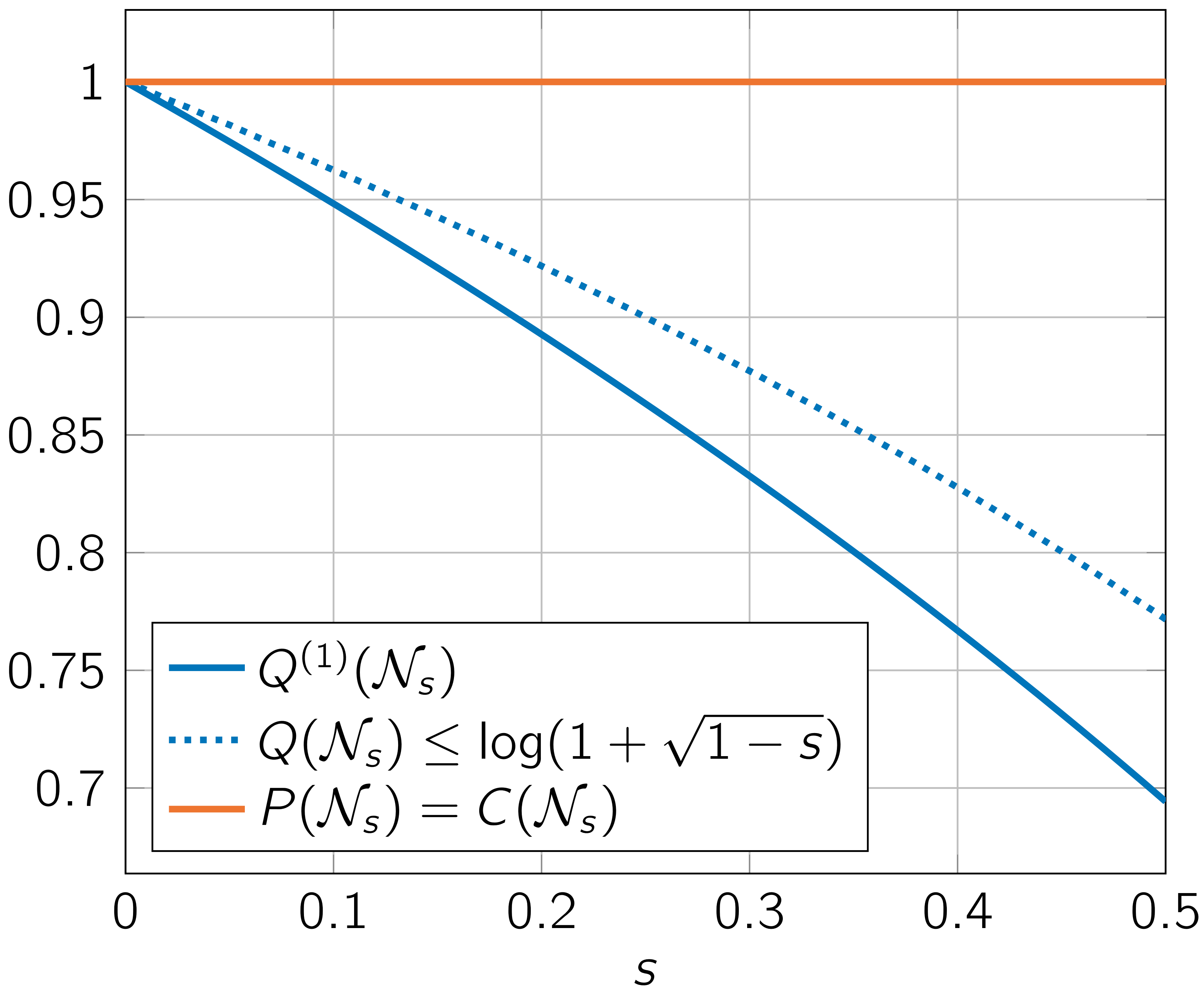
Capacities of the platypus channel:

$$Q^{(1)}(\mathcal{N}_s) \overset{*}{=} Q(\mathcal{N}_s)$$

$$P^{(1)}(\mathcal{N}_s) = P(\mathcal{N}_s)$$

$$C^{(1)}(\mathcal{N}_s) = C(\mathcal{N}_s)$$

*up to the "spin alignment conjecture", more later.



The platypus channel does not belong to any of the known additive channel classes, yet its information quantities are all weakly additive!

Weak additivity of the platypus channel

Private and classical capacity: $P(\mathcal{N}_s) = 1 = C(\mathcal{N}_s)$

$$1 \leq P^{(1)}(\mathcal{N}_s) \leq P(\mathcal{N}_s) \leq C(\mathcal{N}_s) \leq 1$$

Private code with $p_1 = 1/2 = p_2$,
$\rho_1 = |0\rangle\langle 0|$ and $\rho_2 = s|1\rangle\langle 1| + (1-s)|2\rangle\langle 2|$
achieves $P^{(1)}(\mathcal{N}_s) \geq I(X;B) - I(X;E) = 1$.

Strong converse SDP upper bound from [Wang et al. '17] evaluates to 1 (analytically by picking feasible sol).

This also means that $P(\mathcal{N}_s)$ and $C(\mathcal{N}_s)$ have the **strong converse property**!

Spin alignment conjecture

Quantum capacity: $Q^{(1)}(\mathcal{N}_s) = Q(\mathcal{N}_s)$ if the following conjecture is true:

Spin alignment conjecture: Let $n \in \mathbb{N}$, $\{x_M\}_{M\subset[n]}$ a prob. dist., and $Q = \begin{pmatrix} s & 0 \\ 0 & 1-s \end{pmatrix}$.

$$\begin{aligned} \text{min.} \quad & S(\rho) \\ \text{subject to:} \quad & \rho = \sum_{M\subset[n]} x_M \omega_M \otimes Q^{\otimes|M^c|} \\ & \omega_M \geq 0, \mathrm{tr}(\omega_M) = 1. \end{aligned}$$

has the solution $\omega_M = |1\rangle\langle 1|^{\otimes|M|}$ for all $M \subset [n]$.

Solved for $n = 1$ and all s, for $n = 2$ when $s = 1/2$, numerical evidence for $n \leq 6$.

Mohammad Alhejji has a proof for all Rényi entropies of integer order!

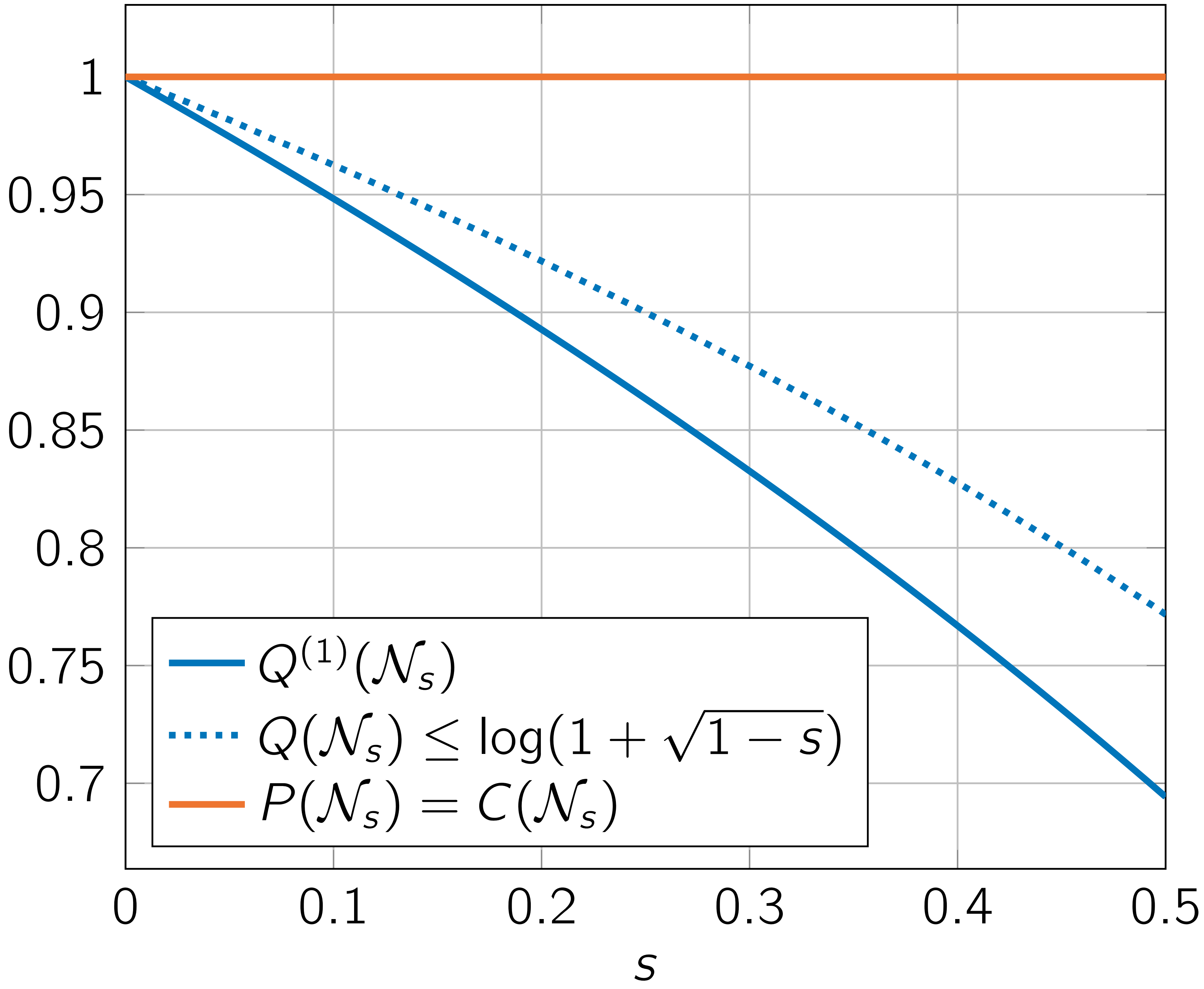
Separation of quantum and private capacity

Even without the spin alignment conjecture, we have an analytical upper bound

$$Q(\mathcal{N}_s) \leq \log(1 + \sqrt{1 - s})$$

obtained by analytically solving an SDP upper bound.

[Werner & Holevo '01, Wang et al. '18]



This proves a **strict separation** $Q(\mathcal{N}_s) < P(\mathcal{N}_s)$ for $s > 0$!

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Strong superadditivity of the platypus channel

Platypus channel is weakly additive but not strongly additive for quantum information transmission:

$$Q^{(1)}(\mathcal{N}_s \otimes \mathcal{K}) > Q^{(1)}(\mathcal{N}_s) + Q^{(1)}(\mathcal{K})$$

where the second channel $\mathcal{K}$ can be one of:

erasure channel $\mathcal{E}_p$, depolarizing channel $\mathcal{D}_p$, qubit Pauli channels, amplitude damping channel $\mathcal{A}_p$, random qubit channels, ...

$\Rightarrow$ $\mathcal{K}$ can be **pretty generic**, and may even have capacity itself!

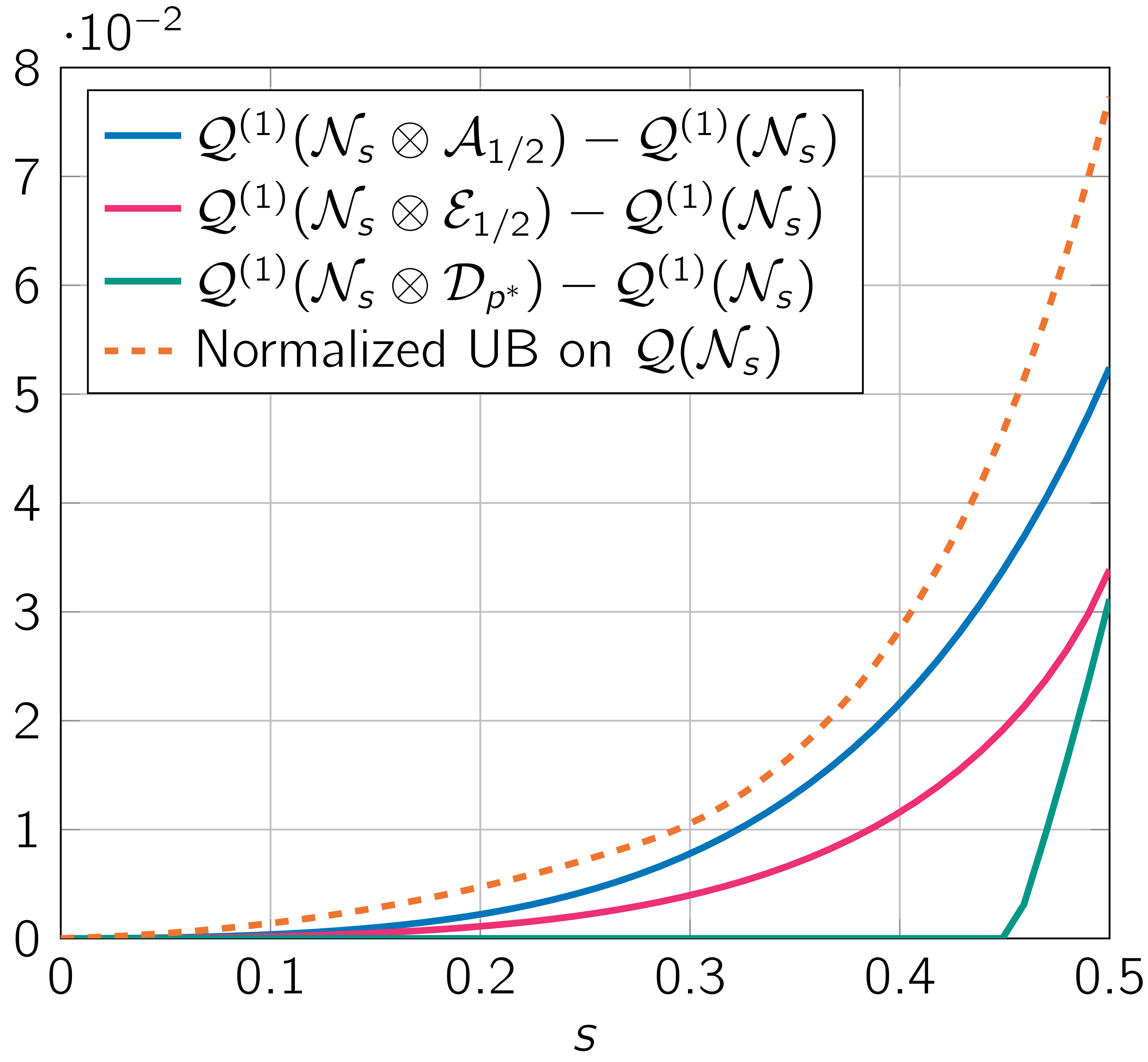
Strong superadditivity of the platypus channel

$$Q^{(1)}(\mathcal{N}_s \otimes \mathcal{K}) > Q^{(1)}(\mathcal{N}_s) + Q^{(1)}(\mathcal{K})$$

Modulo spin alignment conjecture, superadditivity also holds for q. cap.:

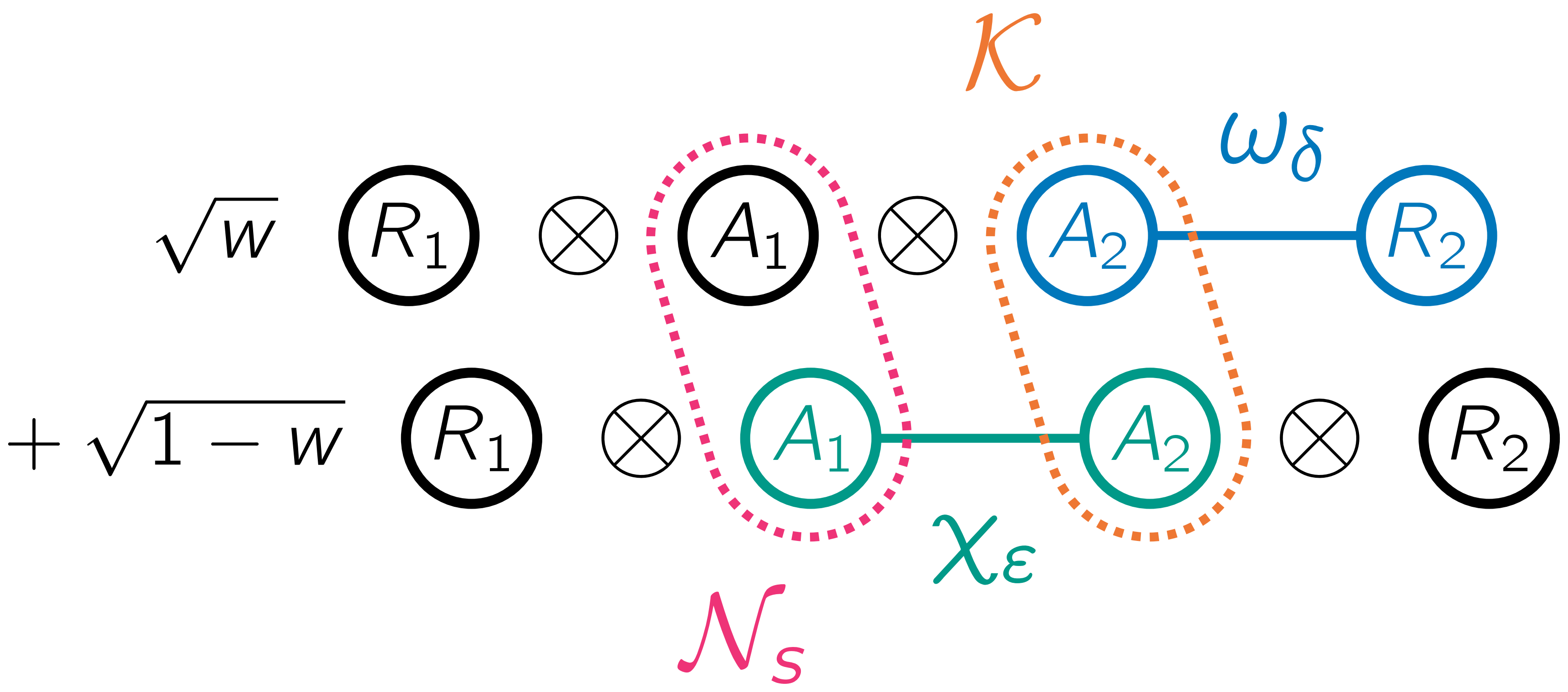
$$Q(\mathcal{N}_s \otimes \mathcal{K}) \overset{*}{>} Q(\mathcal{N}_s) + Q(\mathcal{K})$$

if $\mathcal{K}$ is $\mathcal{E}_x$ or $\mathcal{A}_x$ (since Q is known for these channels).



Strong superadditivity of the platypus channel

A single code ansatz achieves superadditivity for all channels:



$$|\psi\rangle_{R_1R_2A_1A_2} = \sqrt{w}|0\rangle_{R1}|0\rangle_{A_1}|\omega_\delta\rangle_{R_2A_2}$$

$$+\sqrt{1-w}|1\rangle_{R_1}|0\rangle_{R_2}|\chi_\varepsilon\rangle_{A_1A_2}$$

$$|\omega_\delta\rangle_{R_2A_2} = \sqrt{\delta}|00\rangle_{R_2A_2} + \sqrt{1-\delta}|11\rangle_{R_2A_2}$$

$$|\chi_\varepsilon\rangle_{A_1A_2} = \sqrt{\varepsilon}|20\rangle_{A_1A_2} + \sqrt{1-\varepsilon}|11\rangle_{A_1A_2}$$

R_1, R_2: references

A_1: input to $\mathcal{N}_s$

A_2: input to $\mathcal{K}$

Unconditional superadditivity of quantum capacity

We can define a d-**dimensional platypus channel** $\mathcal{M}_d$ $(= \mathcal{N}_{1/2}$ for $d = 3)$.

Spin alignment conjecture: $Q^{(1)}(\mathcal{M}_d) \overset{*}{=} Q(\mathcal{M}_d)$

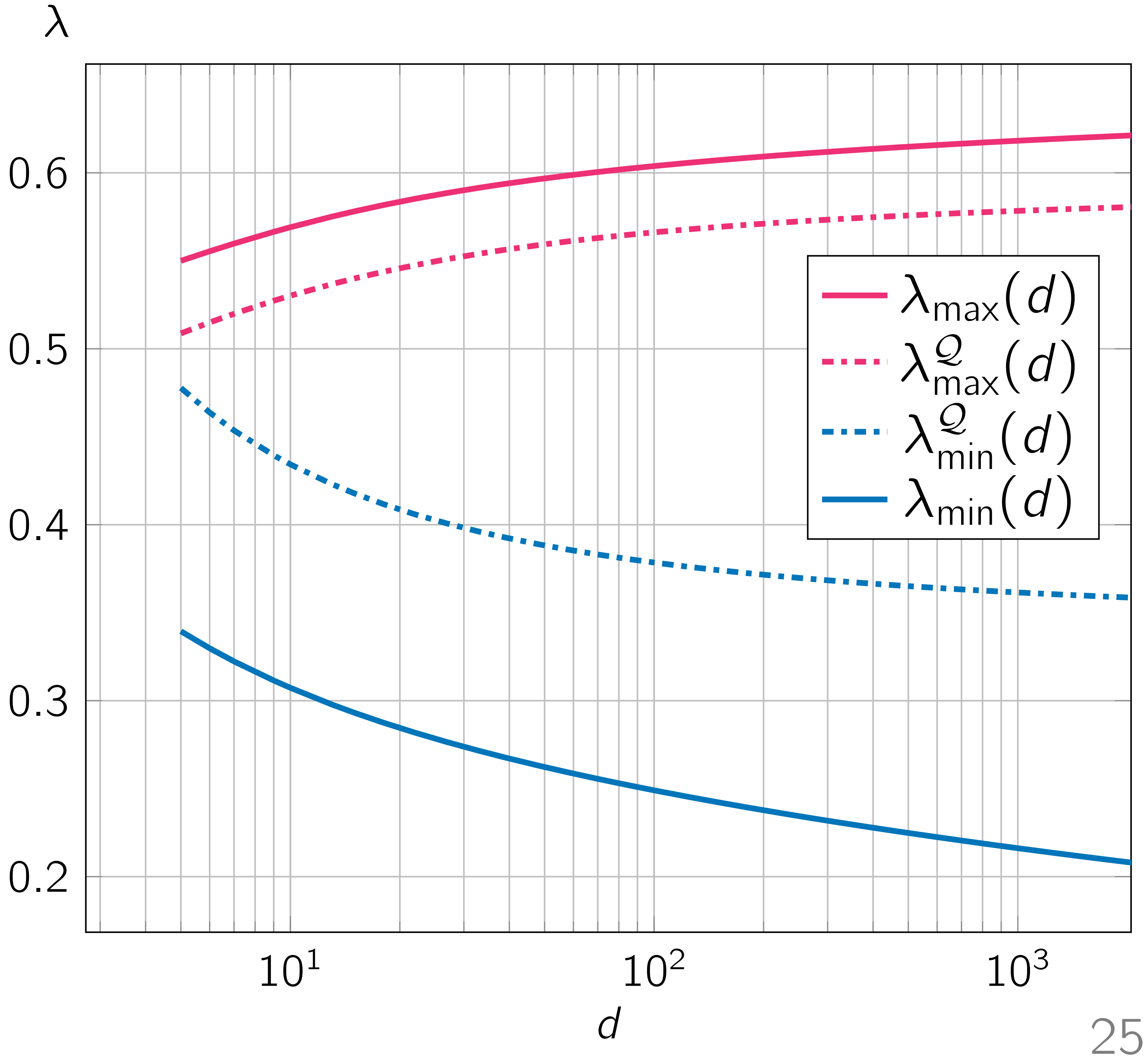
SDP upper bound: $Q(\mathcal{M}_d) \leq \log(1 + \frac{1}{\sqrt{d-1}})$

d-dim. erasure channel $\mathcal{E}_{d,\lambda}$ with capacity

$$Q(\mathcal{E}_{d,\lambda}) = \max\{(1 - 2\lambda) \log d, 0\}.$$

Superadditivity: For $d \geq 5$ and $\lambda = \lambda(d)$,

$$Q(\mathcal{M}_{d+1} \otimes \mathcal{E}_{d,\lambda}) > Q(\mathcal{M}_{d+1}) + Q(\mathcal{E}_{d,\lambda})$$



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Generalization of platypus to more parameters

We can generalize $\mathcal{N}_s$

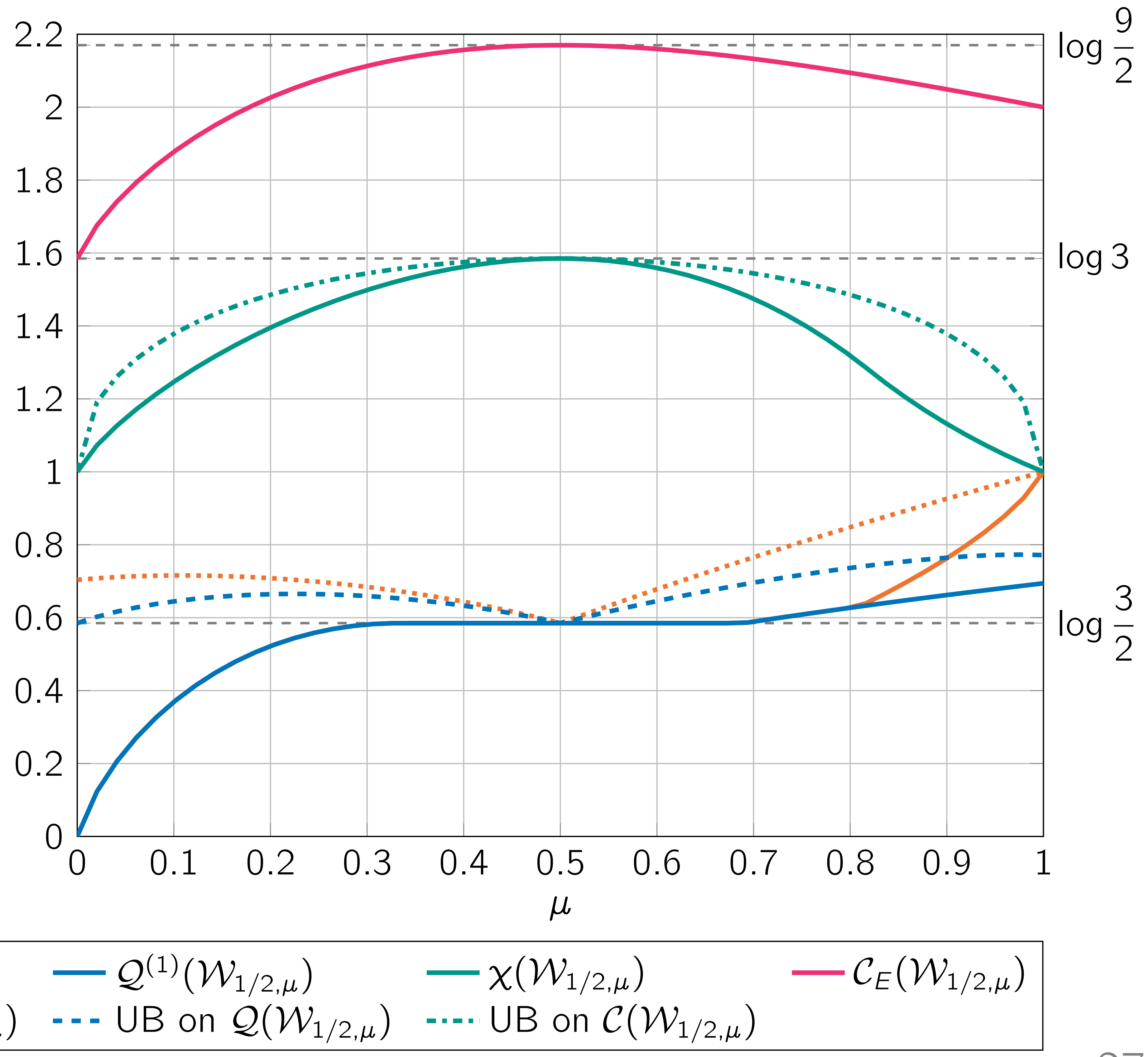
to more parameters, e.g.:

$$\mathcal{W}_{s,\mu} = \mathrm{tr}_E(W_{s,\mu} \cdot W_{s,\mu}^\dagger):$$

$$W_{s,\mu}|0\rangle = \sqrt{s}|00\rangle + \sqrt{1-s}|11\rangle$$

$$W_{s,\mu}|1\rangle = \sqrt{1-\mu}|10\rangle + \sqrt{\mu}|21\rangle$$

$$W_{s,\mu}|2\rangle = \sqrt{\mu}|20\rangle + \sqrt{1-\mu}|01\rangle$$



Previous observations of the platypus in nature

Vikesh defined the qutrit platypus channel in arXiv:2003.10367 to understand so-called log-singularities in the coherent information giving rise to superadditivities.

Even earlier, X. Wang and R. Duan defined a unitarily equivalent channel in arXiv:1608.04508 to study zero-error capacities of channels.

In arXiv:1610.06381, they studied the private and classical capacity of that channel, and also noticed the separation of Q(.) and P(.).

Our work motivates the channel using the stitching construction, provides rigorous analysis of capacities and additivity properties, and generalizes it to higher dimensions.

Conclusion & open problems

The platypus channel is a **weakly additive** channel **of a new type** that **generically** displays **strong non-additivity** for a variety of channels.

The quantum and private capacity are **strictly separated**, and the private and classical capacity satisfy the **strong converse property**.

Strong (non-)additivity for private and classical information?

Can we prove the spin alignment conjecture?

Thank you for your attention!