Beyond IID in Information Theory

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Probing multipartite entanglement through persistent homology

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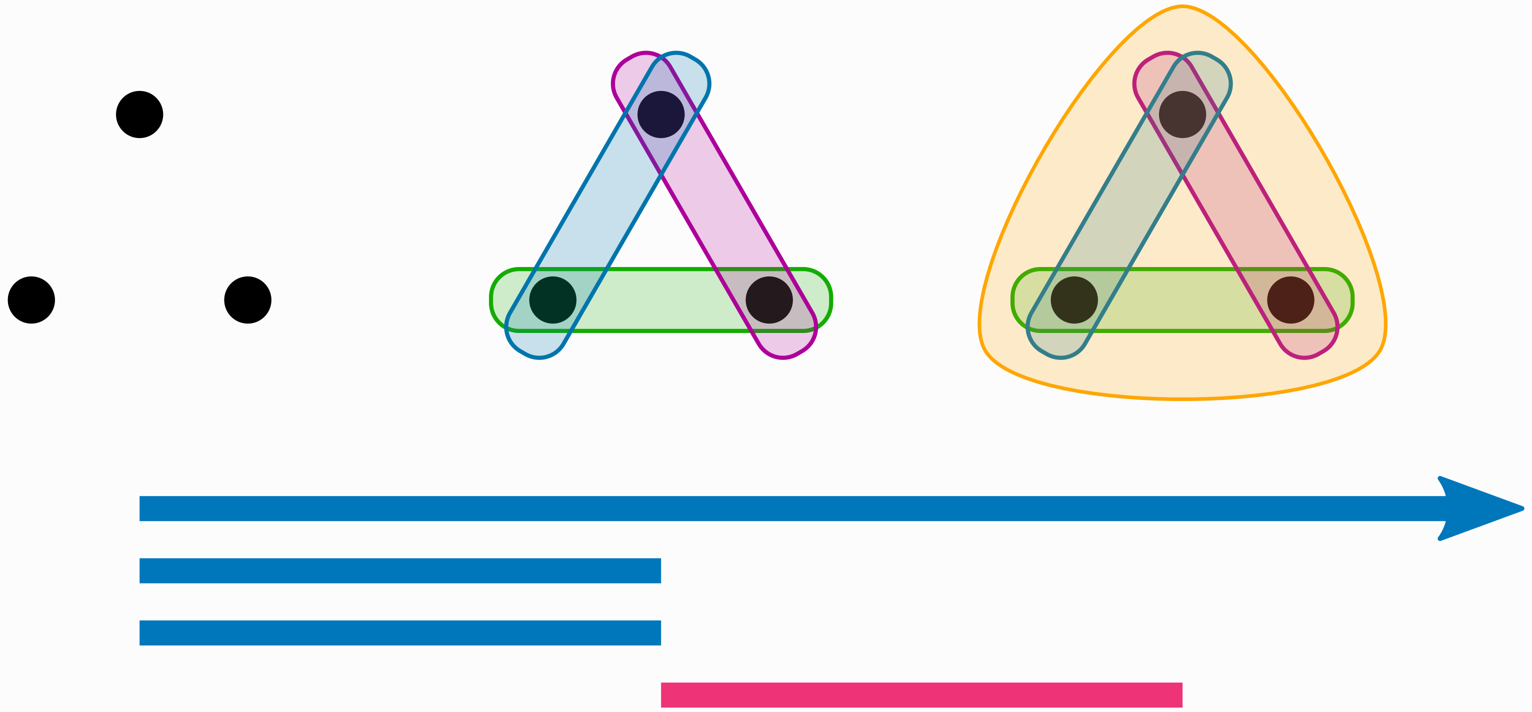

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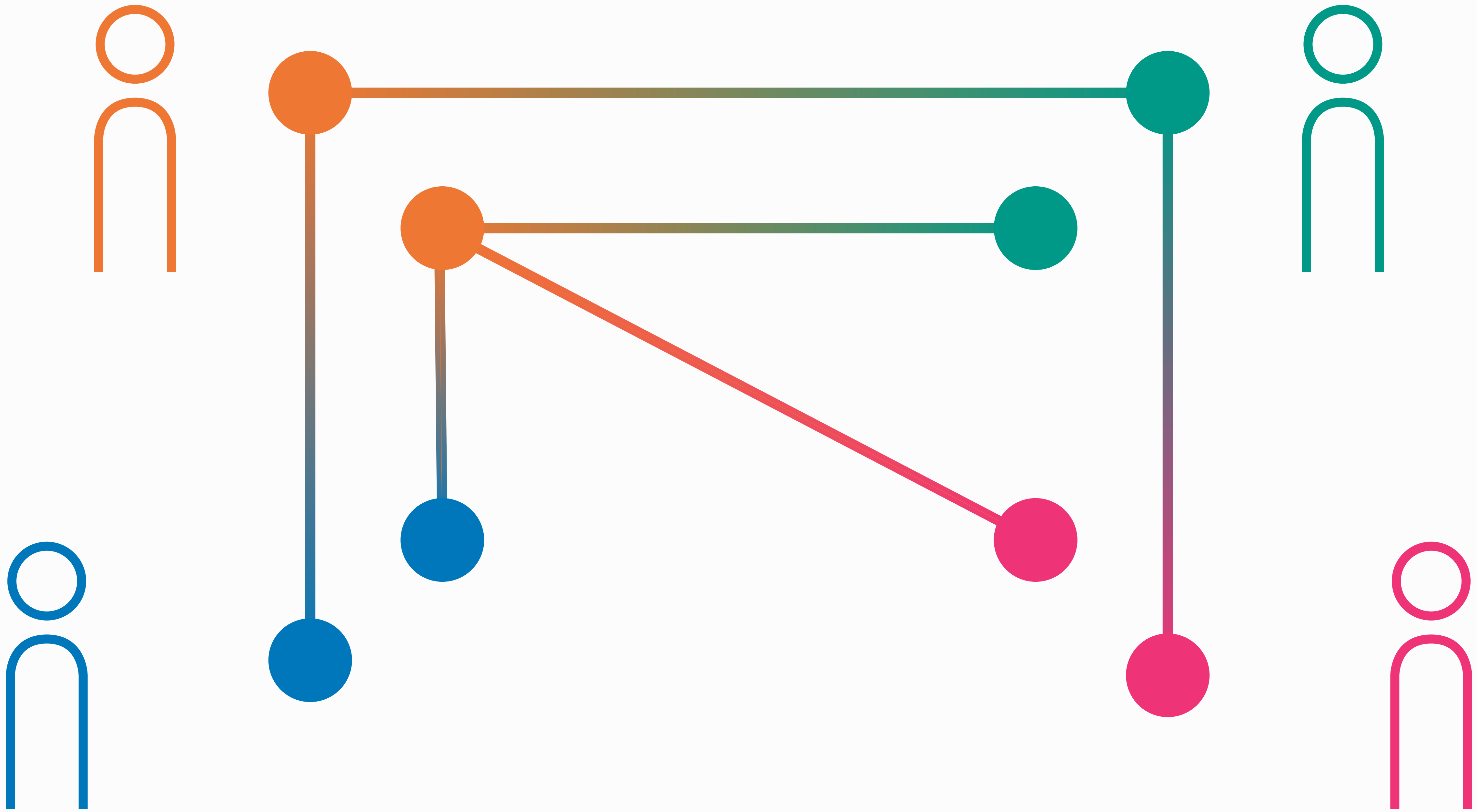
Introduction

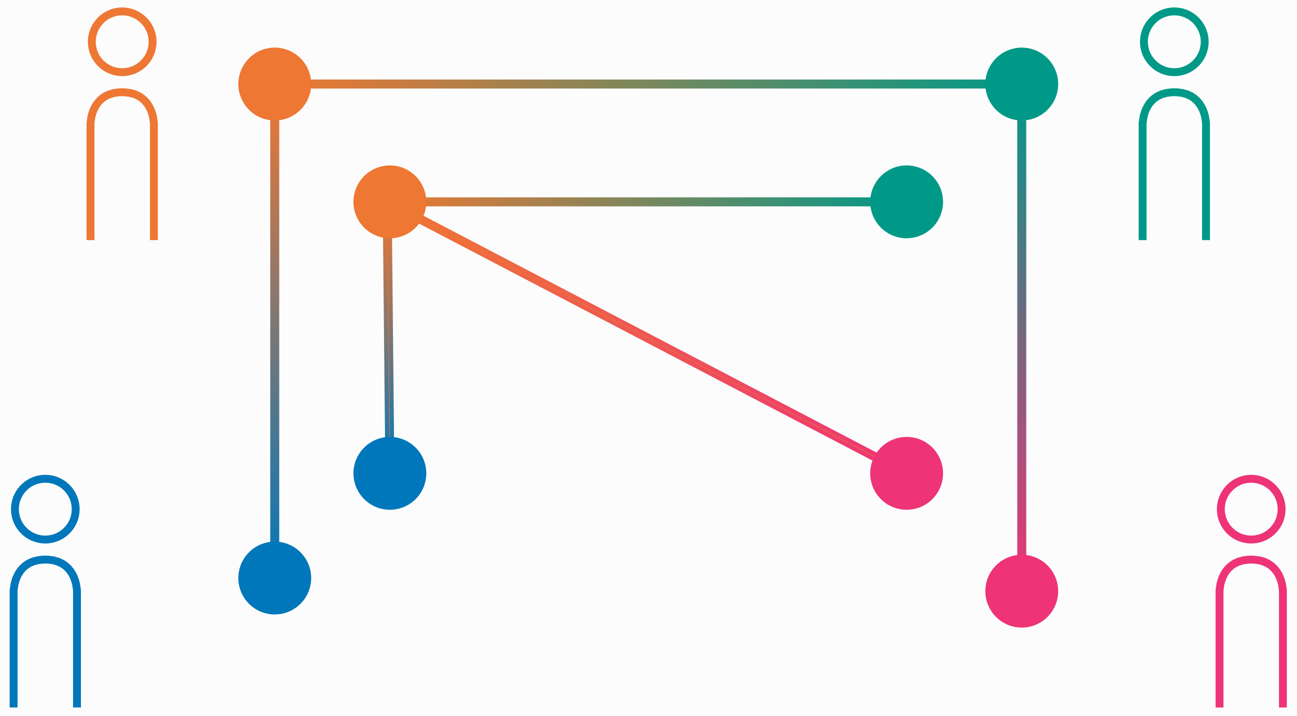
Multipartite quantum systems can exhibit complex correlations that are hard to characterize.

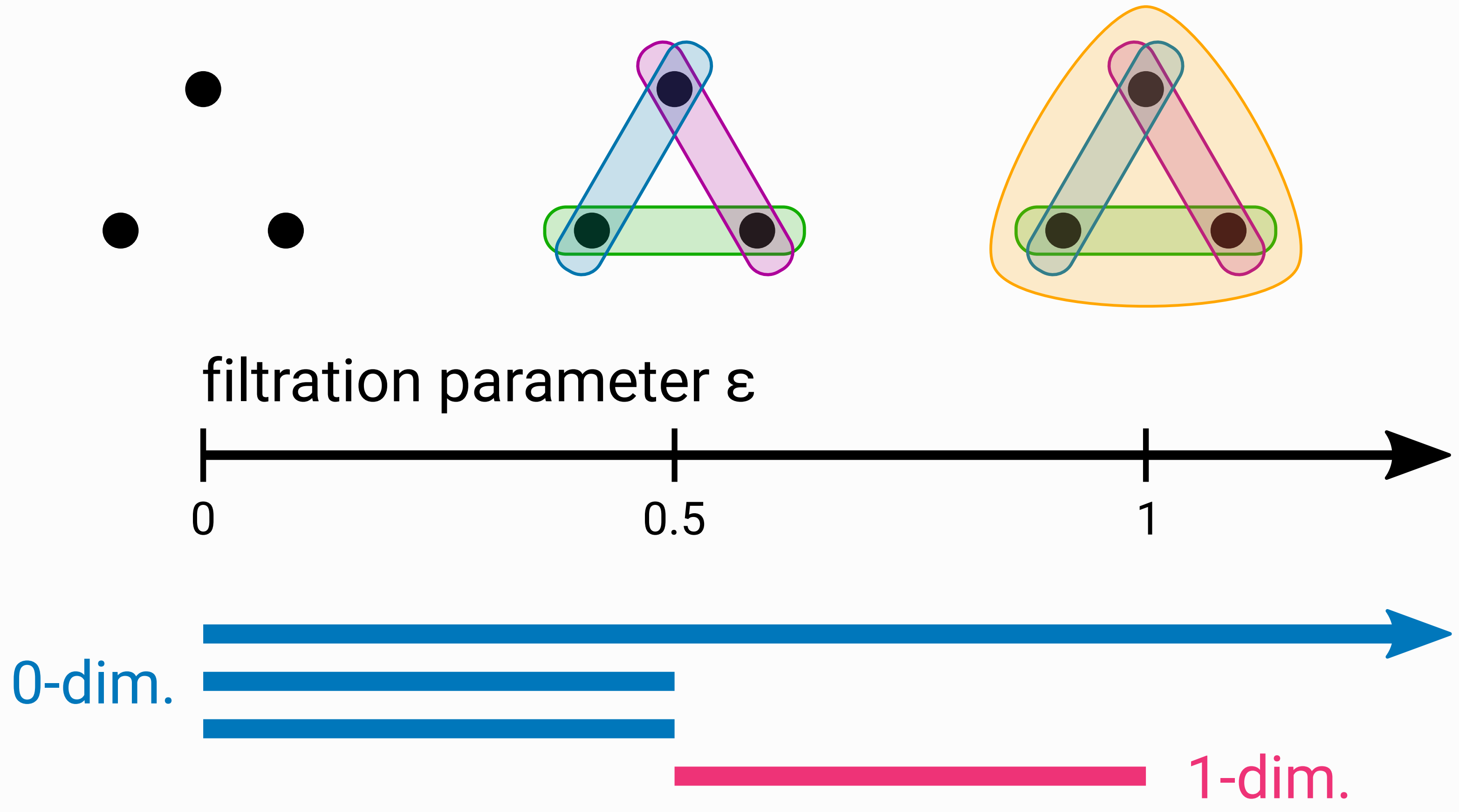


Summary of main results

We propose a **topological approach** that characterizes multipartite entanglement using a tool from topological data analysis called **persistent homology**.

Persistence barcodes visualize the entanglement structure of a multipartite state.





Topological summaries of the persistence complex yield correlation measures and entanglement measures, which assigns them with a **topological interpretation**.

Structure of this talk

- Essentials from entanglement theory
- Persistent homology
- Main results: Correlation functionals as topological summaries
- Future directions of research

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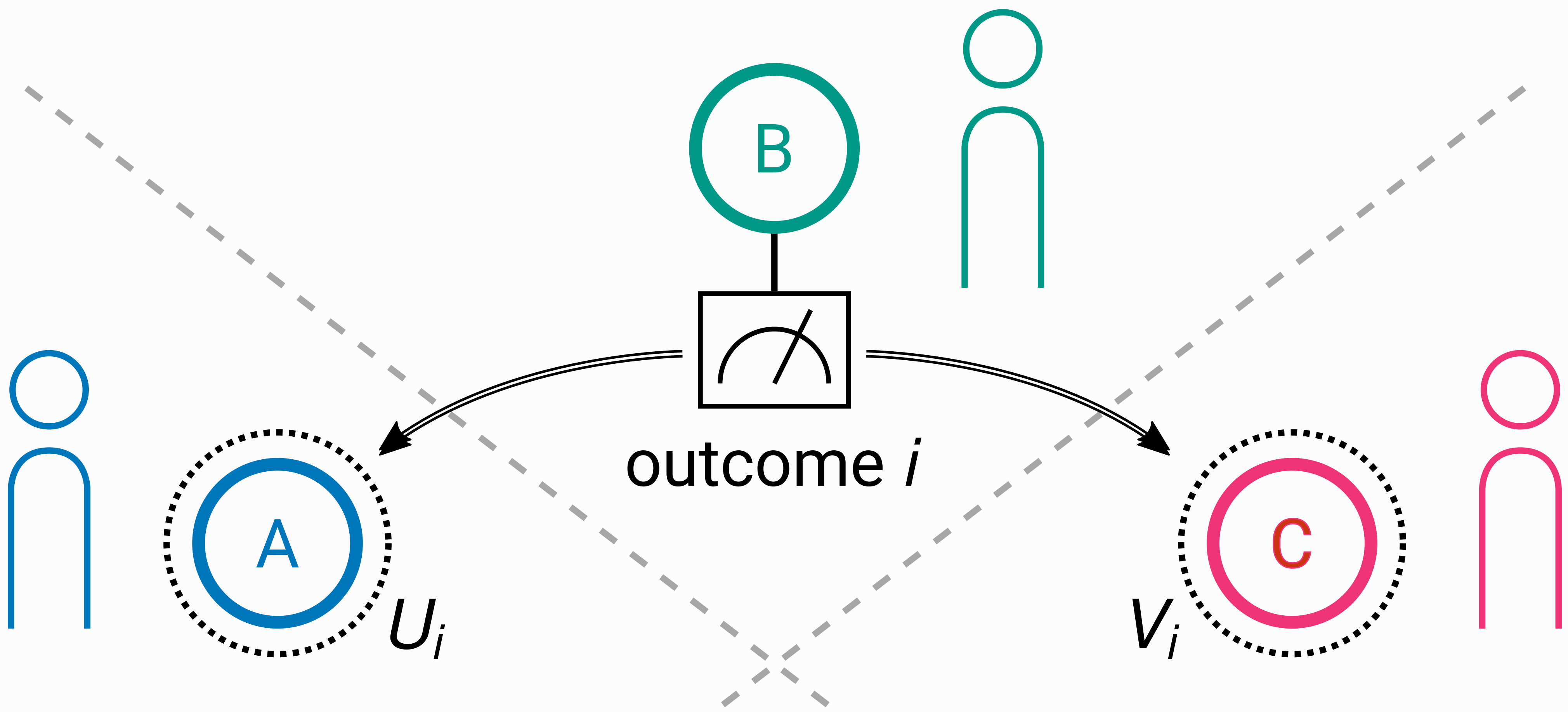
Entanglement and LOCC

A state ρ is **entangled** if it cannot be written as a convex combination of product states.



Local operations and classical communication (LOCC):

Local operations and measurements, whose outcomes can be broadcast to other parties.

Entanglement is a **resource** and cannot be created from scratch using LOCC alone.

Stochastic LOCC

Multipartite LOCC transformations are **notoriously hard** to describe.

[Chitambar et al., Comm. Math. Phys., 2014]

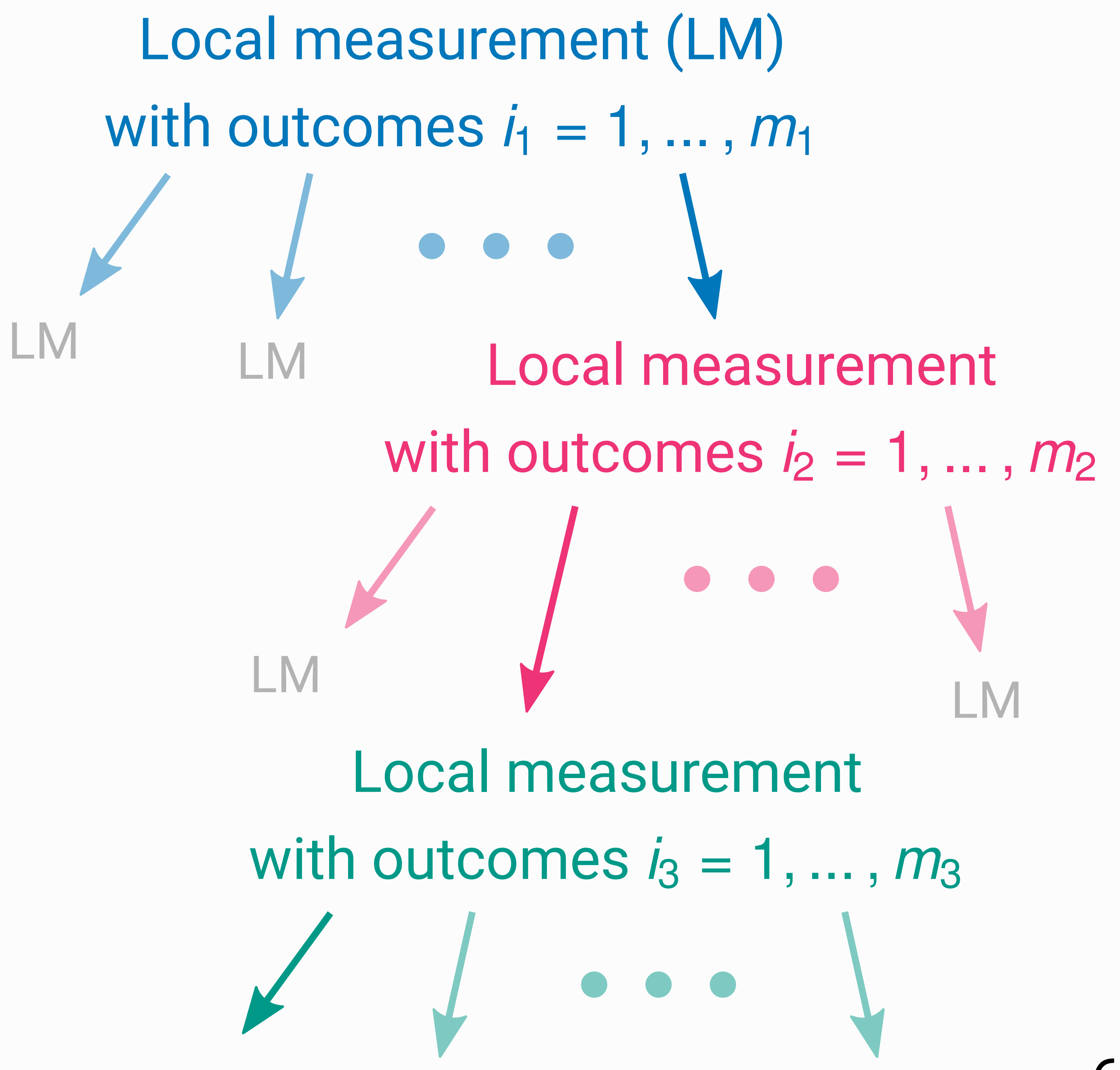
A **stochastic LOCC** (SLOCC) protocol achieves
$\psi \longrightarrow \phi$ with some non-zero **success probability** p.

[Bennett et al., Phys. Rev. A, 2000]

$\psi \xrightarrow{\text{SLOCC}} \phi$ if there are operators A_i and $\lambda \in \mathbb{C}$
such that $(A_1 \otimes \cdots \otimes A_n)|\psi\rangle = \lambda|\phi\rangle$.

ψ, ϕ are **SLOCC-equivalent** if the A_i are invertible.

[Dür et al., Phys. Rev. A, 2000]



SLOCC invariants and entanglement measures

SLOCC-invariant functionals can be used to detect SLOCC-inequivalent states.

[Dür et al., Phys. Rev. A, 2000], [Gour, Wallach, Phys. Rev. Lett., 2013]

In certain situations, SLOCC invariants E are also **entanglement measures**:

$$E(\rho) \geq \sum_i p_i E(\rho_i)$$

for any LOCC protocol mapping ρ to ρ_i with probability p_i, and $E(\sigma) = 0$ for σ separable.

[Verstraete et al., Phys. Rev. A, 2003], [Eltschka et al., Phys. Rev. A, 2012]

Example: n-tangle $\tau_n(\rho) = \mathrm{Tr}\left(\rho\, \sigma_2^{\otimes n} \rho^* \sigma_2^{\otimes n}\right)$

[Wong, Christensen, Phys. Rev. A, 2001]

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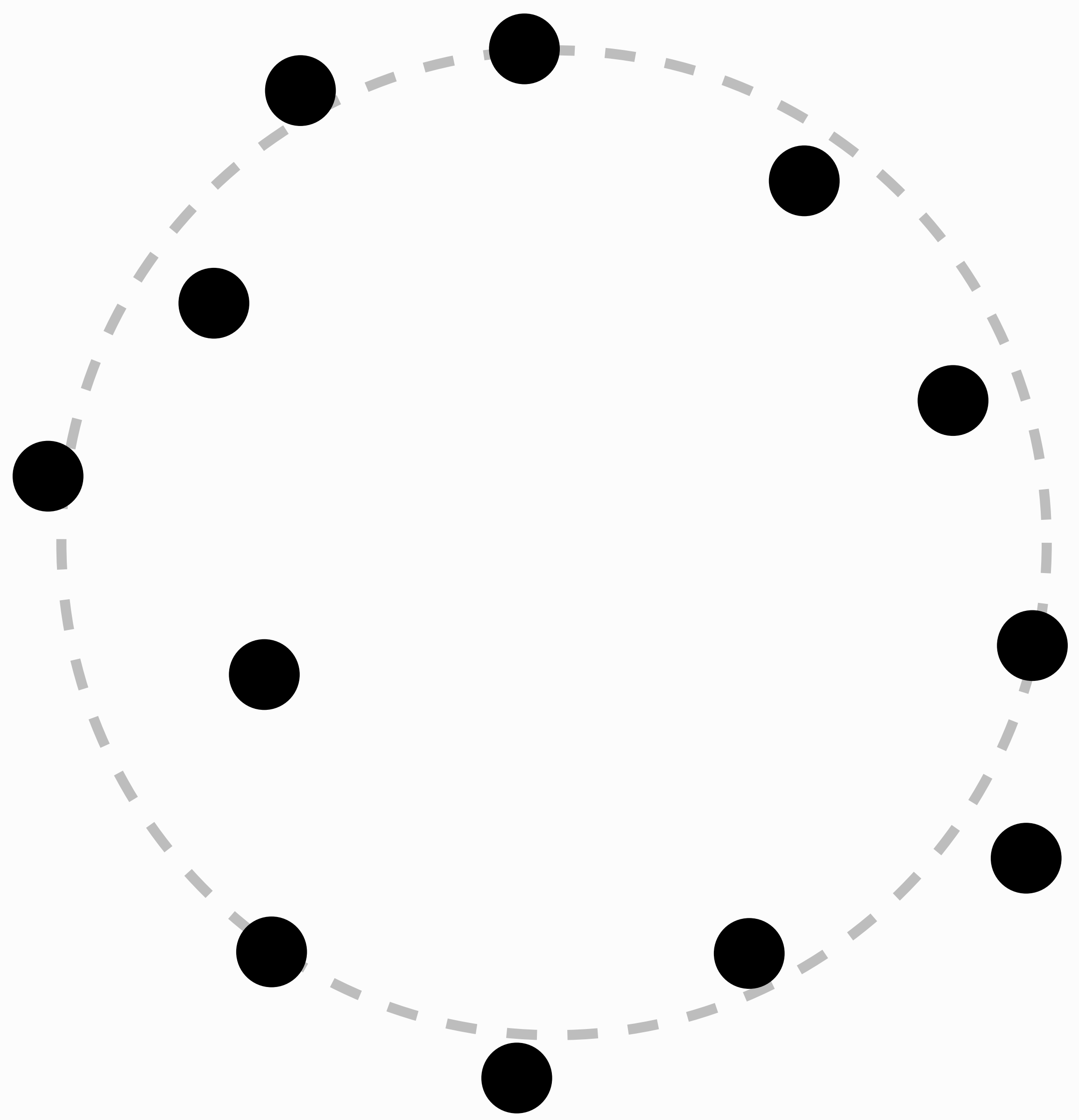
Main idea of persistent homology

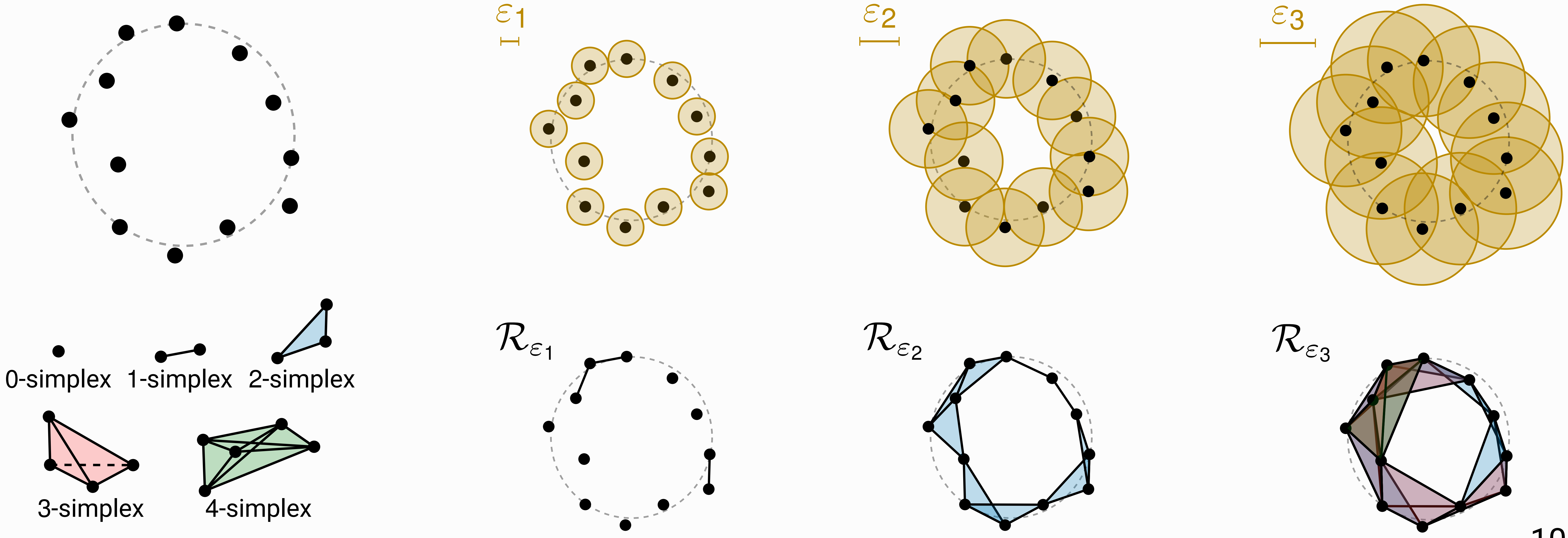
Capture topological features of an underlying source from (noisy) samples.



Persistence complex

Example: Vietoris-Rips complex $\mathcal{R}_\varepsilon$

$k + 1$ vertices form a k-simplex whenever their pairwise distance is less than ε.



Homology groups of simplicial complexes

Chain group $\mathcal{C}_k$: group of k-chains.

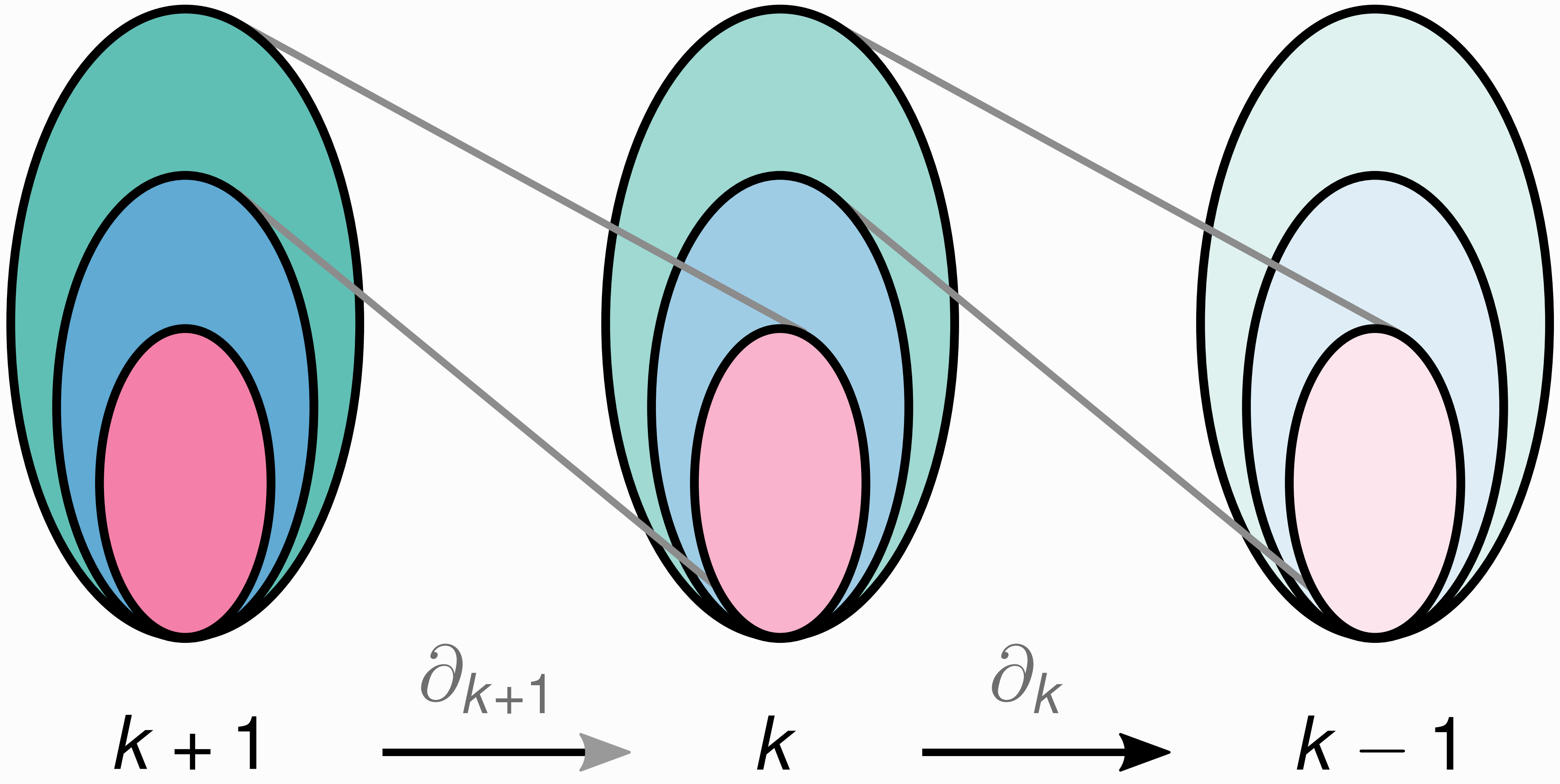
boundary operator $\partial_k : \mathcal{C}_k \to \mathcal{C}_{k-1}$ with $\partial^2 = 0$

Cycle group $Z_k = \ker \partial_k$

$\geq \operatorname{im} \partial_{k+1} = B_k$

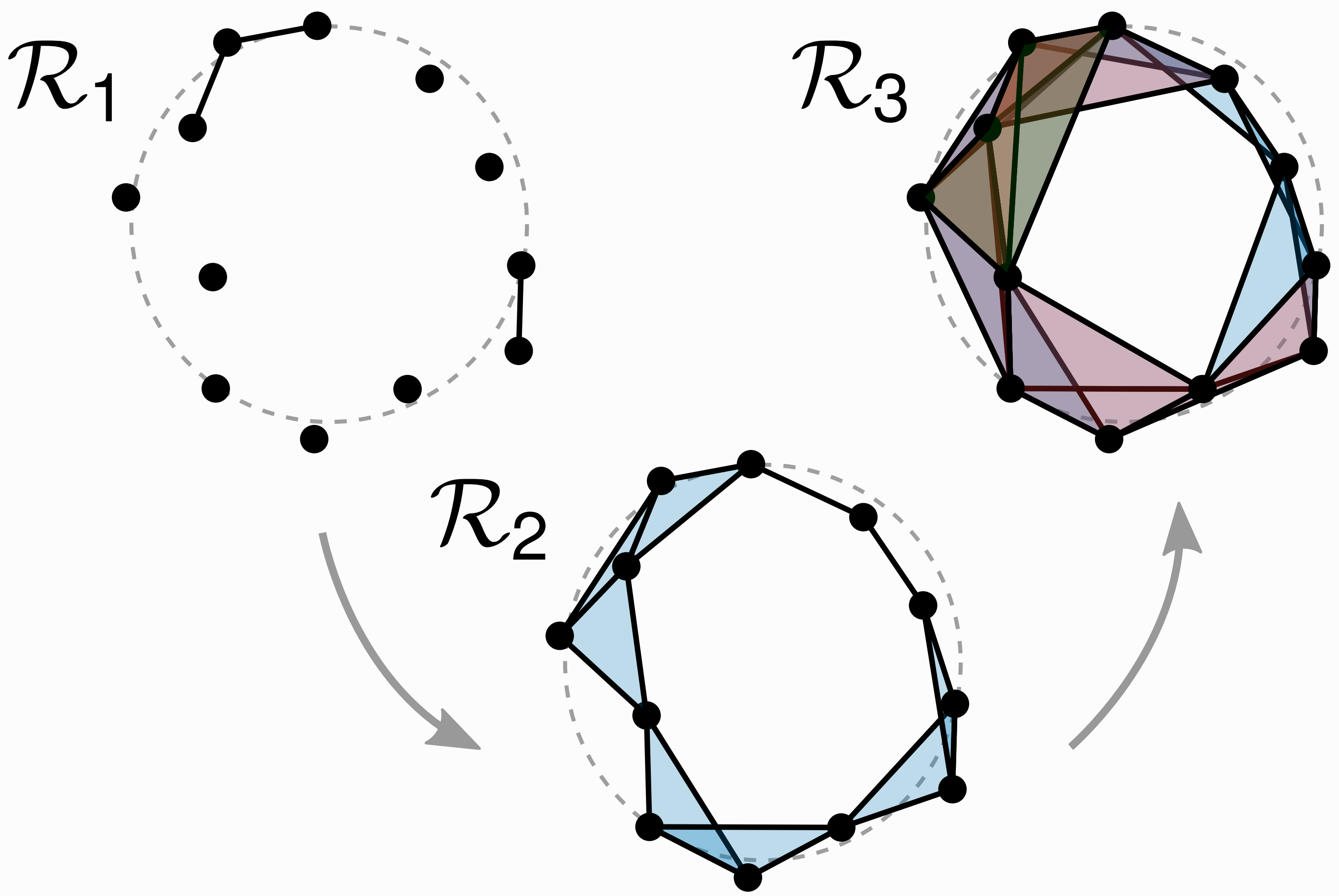
Boundary group $B_k = \operatorname{im} \partial_{k+1}$

k**-th homology group** $H_k = Z_k / B_k$.



Persistence complex

A **persistence complex** is a filtration $\{\mathcal{R}_i \equiv \mathcal{R}_{\varepsilon_i} : \varepsilon_i \in \mathbb{R}\}$ of simplicial complexes with $\mathcal{R}_m \subseteq \mathcal{R}_n$ for $m \leq n$.



The inclusions $R_m \hookrightarrow R_n$ for $m \leq n$ induce homomorphisms $f_k^{m,n} \colon H_k(\mathcal{R}_m) \to H_k(\mathcal{R}_n)$ of homology groups.

Persistence module: Collection of homology groups $(H_*(\mathcal{R}_i))_i$ together with homs. $(f_*^{m,n})_{m \leq n}$.

$(n-m)$**-persistent** k**-th homology group:** $H_k^{m,n} = Z_k(\mathcal{R}_m) \,/\, (B_k(\mathcal{R}_n) \cap Z_k(\mathcal{R}_m)) \cong \operatorname{im} f_k^{m,n}$.

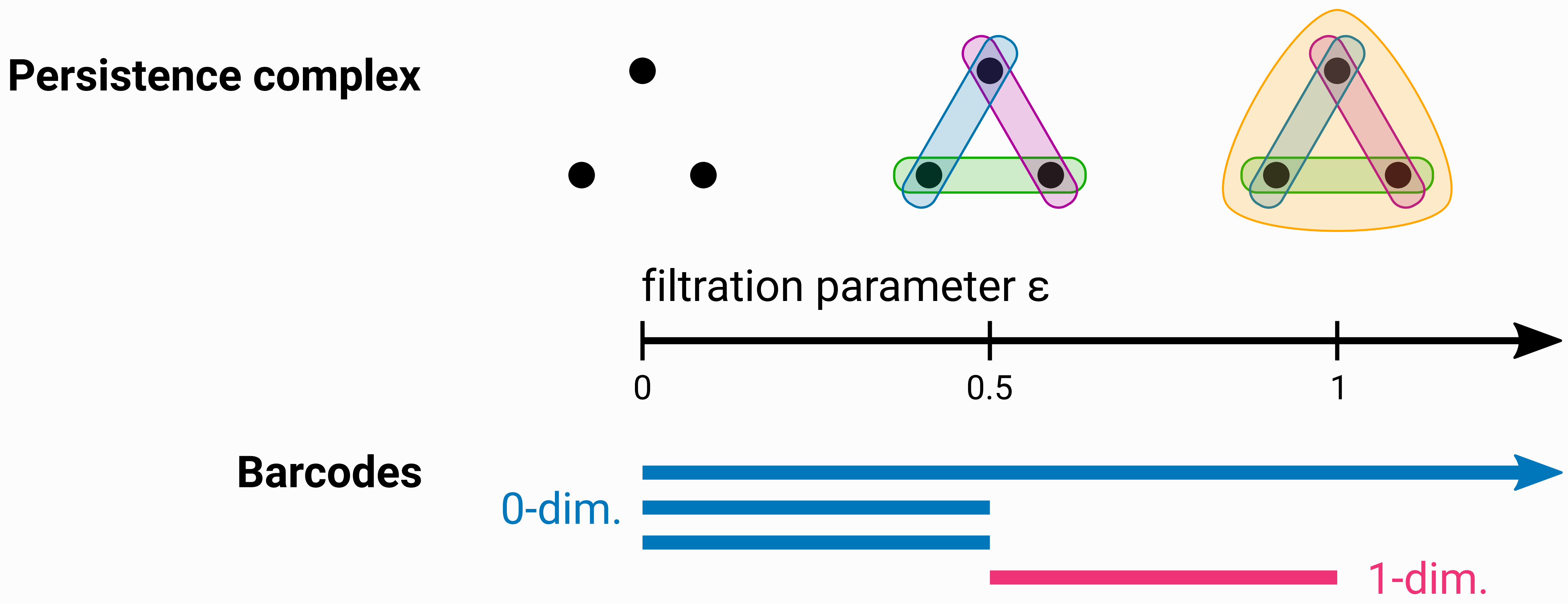
[Zomorodian, Carlsson, Symp. Comp. Geom., 2004]

Barcodes

Define the k**-th Betti number** $\beta_k(\mathcal{R}_\varepsilon)$ of the complex $\mathcal{R}_\varepsilon$ as the rank of $H_k(\mathcal{R}_\varepsilon)$.

The **barcode** of a persistence complex $(\mathcal{R}_\varepsilon)_\varepsilon$ is a collection of stacked intervals.
The x-axis corresponds to ε, and for each k there are $\beta_k(\mathcal{R}_\varepsilon)$ many intervals.

The rank $\beta_k^{m,n}$ = rank $H_k^{m,n}$ equals the number of interval lines spanning the interval $[\varepsilon_m, \varepsilon_n]$.



[Zomorodian, Carlsson, Symp. Comp. Geom., 2004]

Topological summaries

The persistence barcodes encode topological data that can be further summarized:

Topological summaries

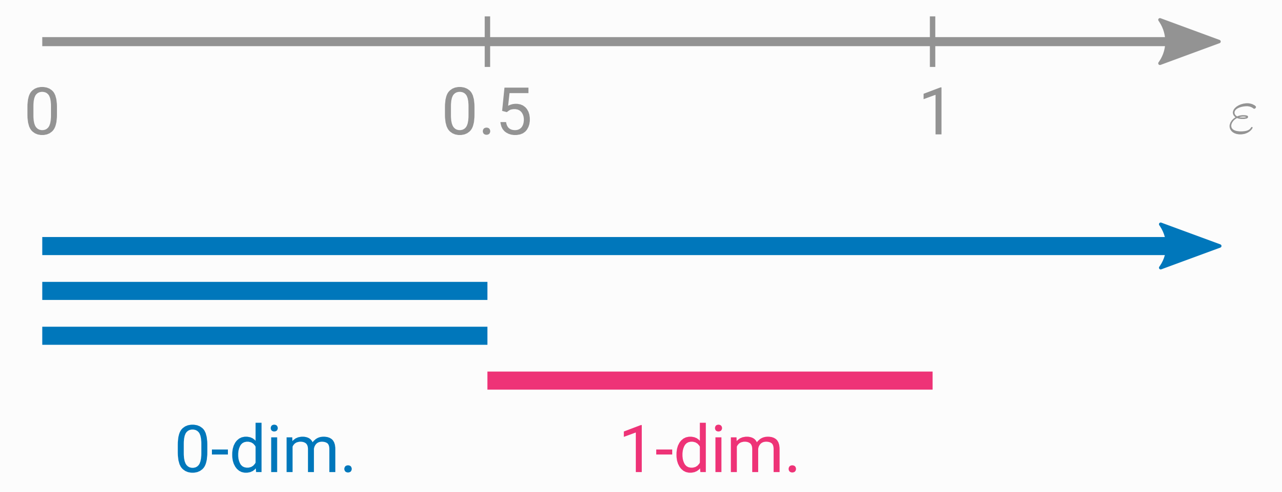
Integrated Betti number $\mathfrak{B}_k(\varepsilon)$ = sum of lengths of k-dim. barcodes in $[0, \varepsilon]$

Integrated Euler characteristic $\mathfrak{X}(\varepsilon) = \sum_k (-1)^k \mathfrak{B}_k(\varepsilon)$

Example: $\mathfrak{B}_0(\infty) = 2 \times (0.5 - 0) = 1$

$\mathfrak{B}_1(\infty) = 1 \times (1 - 0.5) = 0.5$

$\mathfrak{X}(\infty) = \mathfrak{B}_0(\infty) - \mathfrak{B}_1(\infty) = 0.5$



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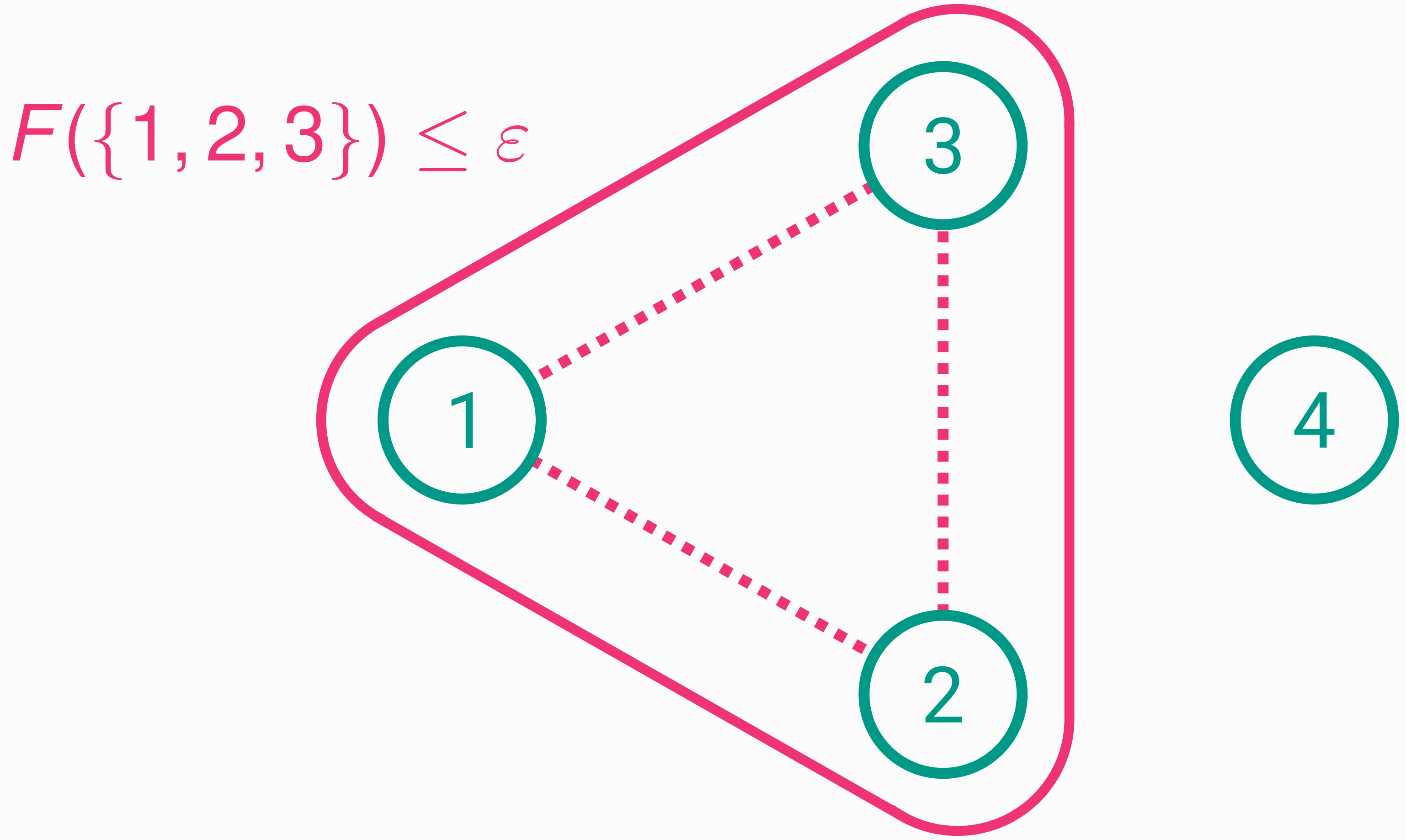
Persistent homology for multipartite systems

We build a persistence complex from a multipartite quantum state ρ on $(\mathbb{C}^d)^{\otimes n}$ as follows:

- Treat local systems of quantum system as vertices of an abstract simplicial complex.
- Choose a functional F defined on marginals $\rho_J = \mathrm{Tr}_{J^c}\,\psi$ with $J \subseteq [n]$ and set $F(J) = F(\rho_J)$.
- For a fixed filtration parameter $\varepsilon \in \mathbb{R}_+$ we add a simplex $J \subseteq [n]$ to Δ if $F(J) \leq \varepsilon$.



- This defines a valid complex $\mathcal{R}_\varepsilon$ provided that F is **monotonic with respect to taking subsets**:

$$F(J) \leq F(K) \text{ for } J \subseteq K \subseteq [n]$$

- The k-simplices of $\mathcal{R}_\varepsilon$ are the simplicies $J \subseteq [n]$ with $|J| = k + 1$ and $F(J) \leq \varepsilon$.

Choice of functional

For $q \geq 1$ and $J \subseteq [n]$ we define the **Tsallis entropy** $S_q(J) = \frac{1}{1-q}\left(\operatorname{Tr}\rho_J^q - 1\right)$ of $\rho_J = \operatorname{Tr}_{J^c}\rho$.

Defining the persistence complex

We choose as $F\colon 2^{[n]} \to \mathbb{R}_+$ the q**-deformed total correlation**

$$C_q(J) = \sum\nolimits_{v\in J} S_q(v) - S_q(J)$$

Subset monotonicity follows from subadditivity of Tsallis entropy. [Audenaert, J. Math. Phys, 2007]

Operational interpretation for q=1: [Groisman et al., , Phys. Rev. A, 2005]

Total amount of local noise needed to decouple J from the rest of the systems.

Interaction information as topological summary

Main result 1

For the persistence module defined in terms of the *q*-deformed total correlation,

the integrated Euler characteristic $\mathfrak{X}(\infty) = \sum_k (-1)^k \mathfrak{B}_k(\infty)$ equals

$$\mathcal{I}_q = \sum_{J \subseteq [n]} (-1)^{|J|-1} S_q(J),$$

the ***q*-deformed interaction information**.

For $q = 1$, the interaction information is an n-partite generalization of mutual information ($n = 2$) and tripartite information ($n = 3$).

Special case $q = 2$ gives n-tangle

Main result 2

For the persistence module of an n-qubit state $|\psi\rangle$ defined in terms of the 2-deformed total correlation, the IEC $\mathfrak{X}(\infty)$ equals the n-tangle $\tau_n = |\langle\psi|\sigma_2^{\otimes n}|\psi^*\rangle|^2$:

$$\mathfrak{X}(\infty) = \mathcal{I}_2 = \tau_n$$

The proof relies on the n-qubit Bloch vector coefficients $Q_{(i_1,\dots,i_n)} = \langle\psi|\sigma_{i_1} \otimes \cdots \otimes \sigma_{i_n}|\psi\rangle$, and writing the n-tangle as $\tau_n(\psi) = \sum_{J\subseteq[n]} (-1)^{|J|-1} S_2(J)$.

[Jaeger et al., Phys. Rev. A, 2003]

n-tangle as a topological summary

The n-tangle is an SLOCC invariant as well as an entanglement measure.

[Wong, Christensen, Phys. Rev. A, 2001]

Our result

Integrated Euler characteristic of persistence complex $(\mathcal{R}_\varepsilon)_\varepsilon$ is a **topological summary** giving information about **multipartite entanglement** in ψ.

This answers a previously raised question about a topological interpretation of the n-tangle.

[Eltschka, Siewert, Quantum, 2018]

SLOCC-inequivalent states with equal *n*-tangle

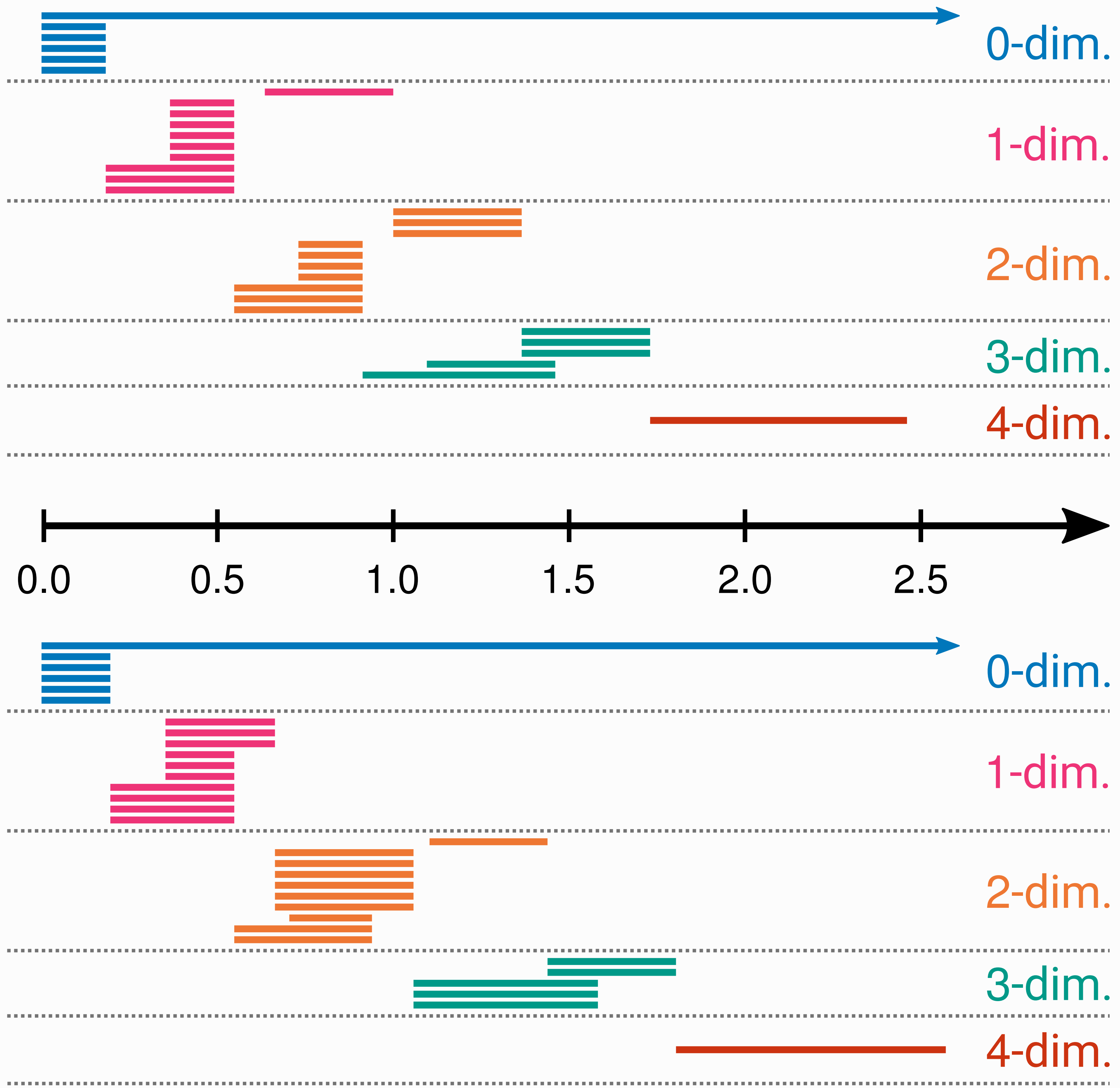
$$|\chi_4\rangle \propto \frac{3}{4}|111111\rangle + \frac{3}{4}|111100\rangle + \frac{4}{3}|000010\rangle + \frac{4}{3}|000001\rangle$$

$$\tau_n(\Xi_4) = 0 = \tau_n(\Xi_5)$$

Ξ_4 and Ξ_5 are **SLOCC-inequivalent**.

$$|\chi_5\rangle \propto \frac{3}{4}|111111\rangle + \frac{3}{4}|111000\rangle + \frac{3}{4}|000100\rangle + \frac{3}{4}|000010\rangle + \frac{3}{4}|000001\rangle$$

[Osterloh, Siewert, Int. J. Quant. Inf., 2006]



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Barcodes and entanglement properties

n-tangle $\longleftrightarrow$ integrated Euler characteristic (IEC) from 2-total correlation

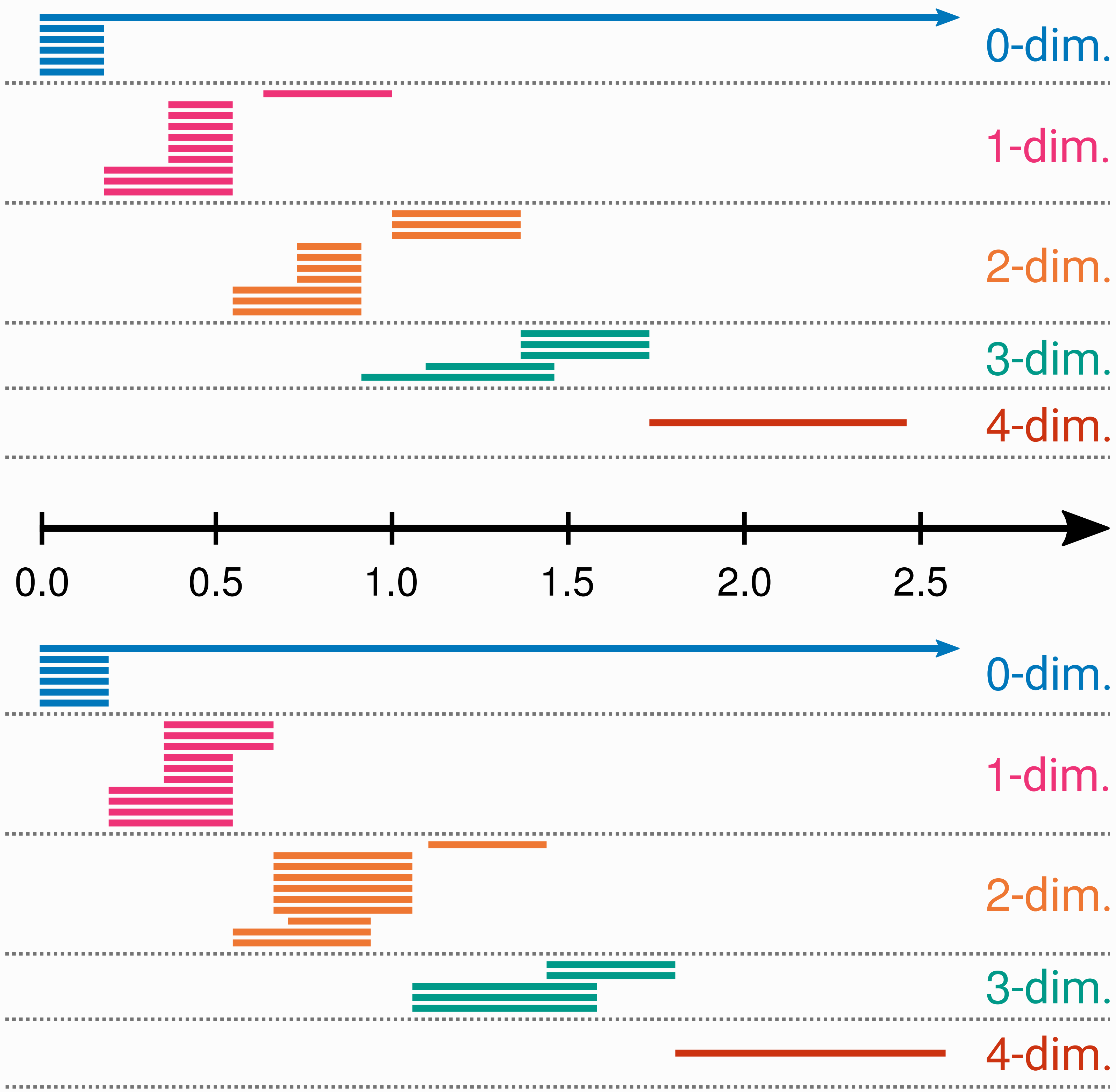
IEC $\longleftrightarrow$ function of persistence barcodes

Questions

Can we use persistence barcodes to distinguish SLOCC classes?

Can we attach an operational meaning to the barcodes themselves?

What other entanglement measures can be expressed as topological summaries?



Generalized divergences and resource theories

Total correlation: $C(J) = \sum_{x \in J} S(x) - S(J) = D\Big(\rho_J \,\Big\|\, \bigotimes_{x \in J} \rho_x\Big) = \min_{\{\sigma_x\}_{x \in J}} D\Big(\rho_J \,\Big\|\, \bigotimes_{x \in J} \sigma_x\Big)$

relative entropy distance from set of **uncorrelated states**

Subset-monotonicity $C(J) \leq C(K)$ for $J \subseteq K$ follows from **data-processing**.

Idea

Choose a **generalized divergence** (functional $D(\cdot\|\cdot)$ satisfying data-processing) and define functional as generalized divergence distance to free set of some resource theory.

Can we express monotones in this resource theory as topological quantities as well?

Conclusion

We define a **persistence complex** for a multipartite quantum state
in terms of a functional quantifying the **correlations** within subsets of the system,
and compute topological summaries such as the **integrated Euler characteristic**.

For a special choice of functional, the integrated Euler characteristic of this persistence complex equals an **entanglement monotone** and thus gives **operational and topological** information about the **multipartite entanglement** structure.

Not mentioned in this talk:
We also reveal a connection to entropy inequalities by studying **relative homology**,
which is intimately connected to **strong subadditivity**.

Thank you for your attention!