Tutte Colloquium July 10, 2020

Symmetries and asymptotics of port-based teleportation

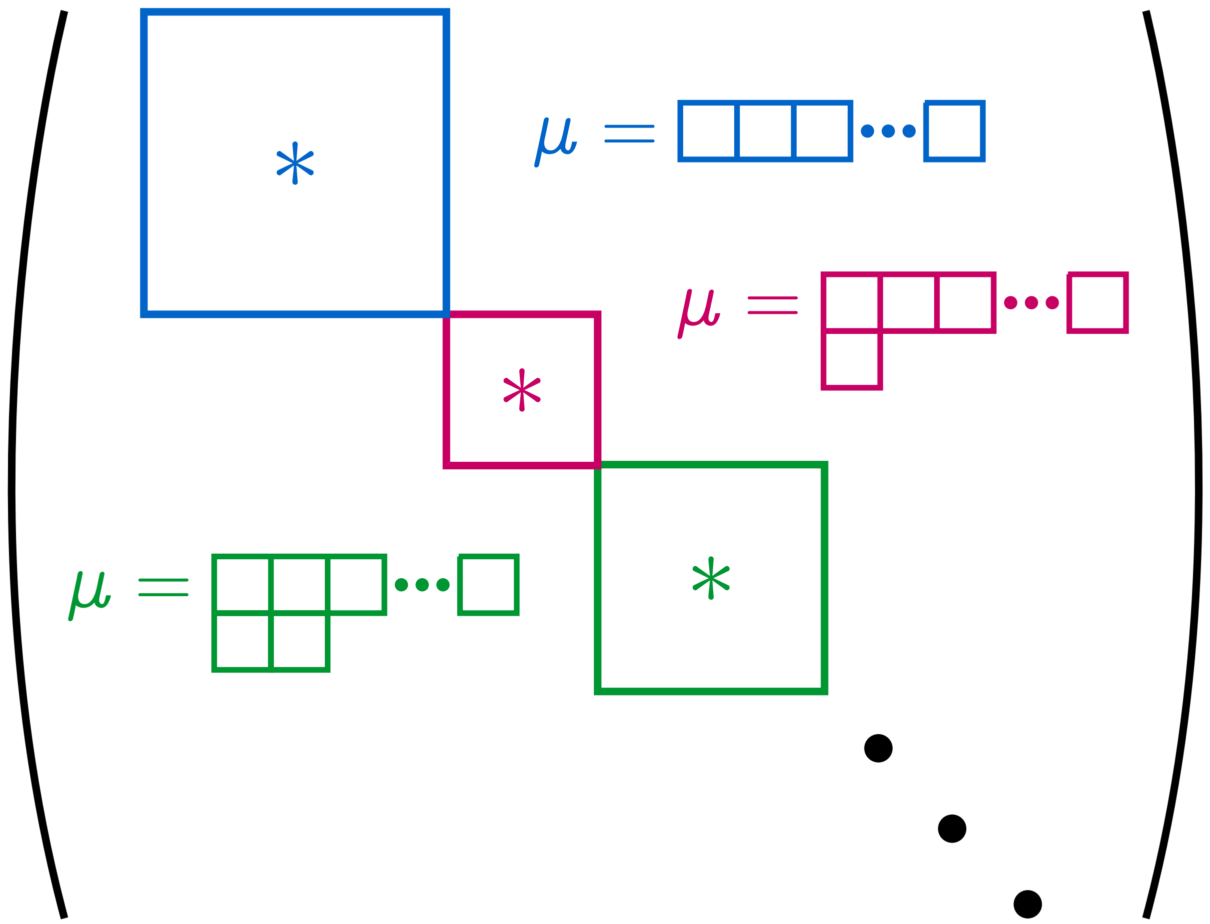
Felix Leditzky

IQC, University of Waterloo

Perimeter Institute for Theoretical Physics

partly based on arXiv:1809.10751

(with M. Christandl, C. Majenz, G. Smith, F. Speelman, M. Walter)



Entanglement and teleportation

Photo: Abdus Salam ICTP Dirac Medal Award

"An entangled state describes the complete knowledge of the whole without knowing the state of any one part."

- Charles H. Bennett

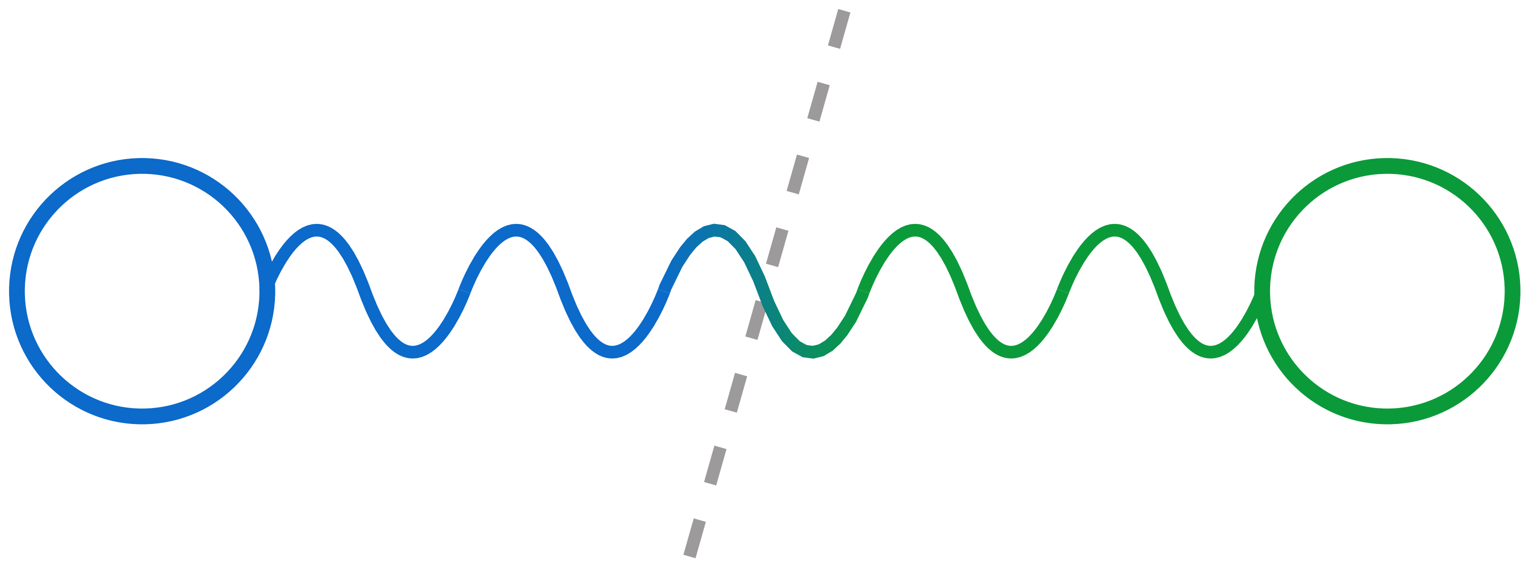
(Shannon Award 2020)

Entanglement and teleportation

Entanglement: strong form of **non-local correlation** between separated systems.

Incredibly useful for quantum information-processing

when used together with other resources.

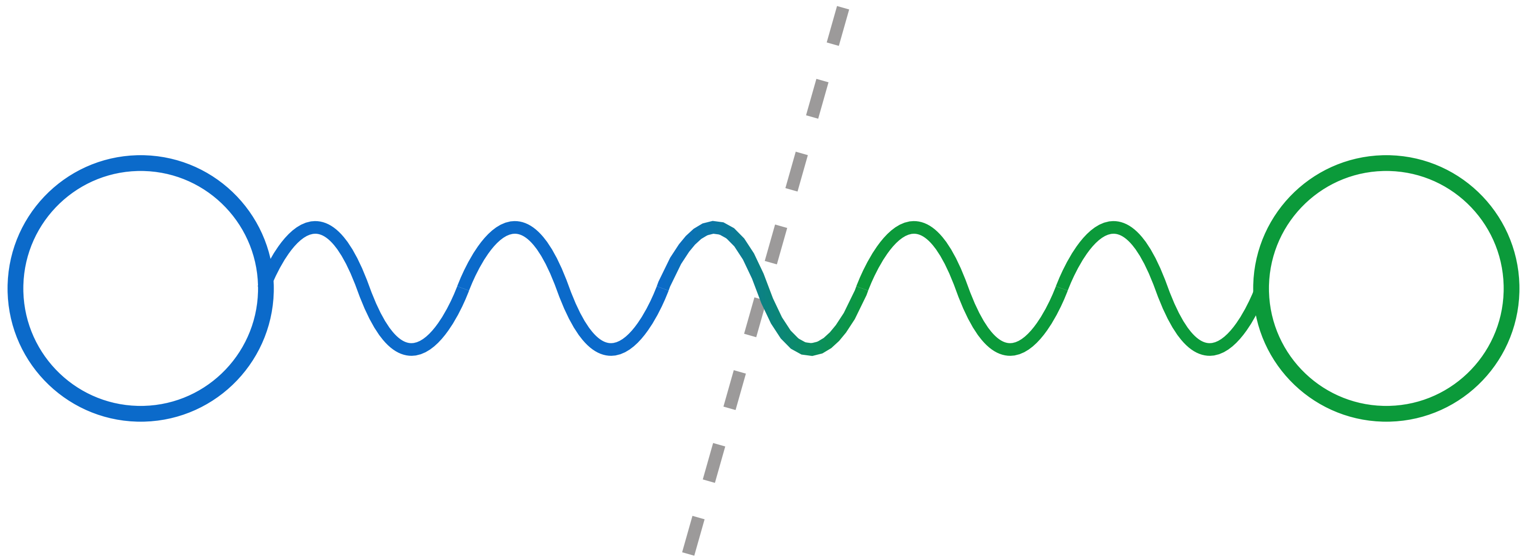


Entanglement and teleportation

Breakthrough result in 1993: **Quantum teleportation**

Bennett et al. (see image) realized that correlation in an entangled state and classical communication can be used to **teleport an unknown quantum state**.

(top, left) Richard Jozsa, William K. Wootters, Charles H. Bennett. (bottom, left) Gilles Brassard, Claude Crépeau, Asher Peres. Photo: André Berthiaume.



[Bennett et al. '93]

Table of contents

Quantum information 101

→ Quantum systems are modeled by (finite-dimensional) Hilbert spaces $\mathcal{H}$.

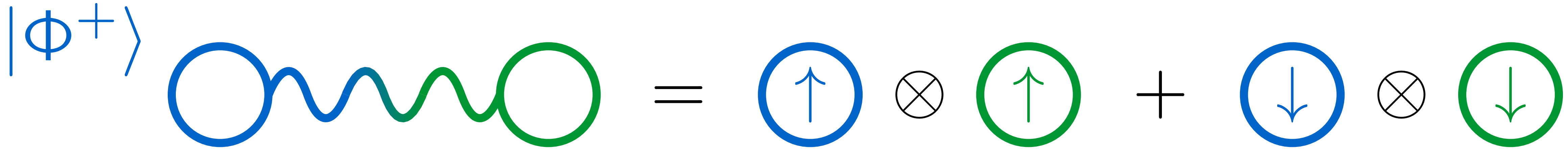
→ A **pure state** is a normalized vector $|\psi\rangle \in \mathcal{H}$. $\big[\langle\psi| = (|\psi\rangle)^\dagger\big]$

→ A **mixed state** is a probabilistic mixture $\sum_i p_i|\psi_i\rangle\langle\psi_i|$ of pure states.
Alternatively, mixed states are linear PSD operators with unit trace.

→ Composite quantum systems AB "live" on a tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$.
A (pure) state is **entangled** if it cannot be written as a product state.

Quantum information 102

→ **Maximally entangled state:** $|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\uparrow\rangle_B + |\downarrow\rangle_A \otimes |\downarrow\rangle_B)$



→ For bipartite states ρ_{AB} the **marginal** state is obtained by applying the **partial trace** operation: $\mathrm{Tr}[\mathrm{Tr}_B(\rho_{AB})X_A] = \mathrm{Tr}[\rho_{AB}(X_A \otimes \mathbb{1}_B)]$ for all X_A.

→ The marginal states of the maximally entangled state are **completely mixed**:

$$\mathrm{Tr}_A\, \Phi^+_{AB} = \frac{1}{2}\mathbb{1} = \mathrm{Tr}_B\, \Phi^+_{AB}.$$

→ Remember Charlie: *"An entangled state describes the complete knowledge of the whole without knowing the state of any one part."*

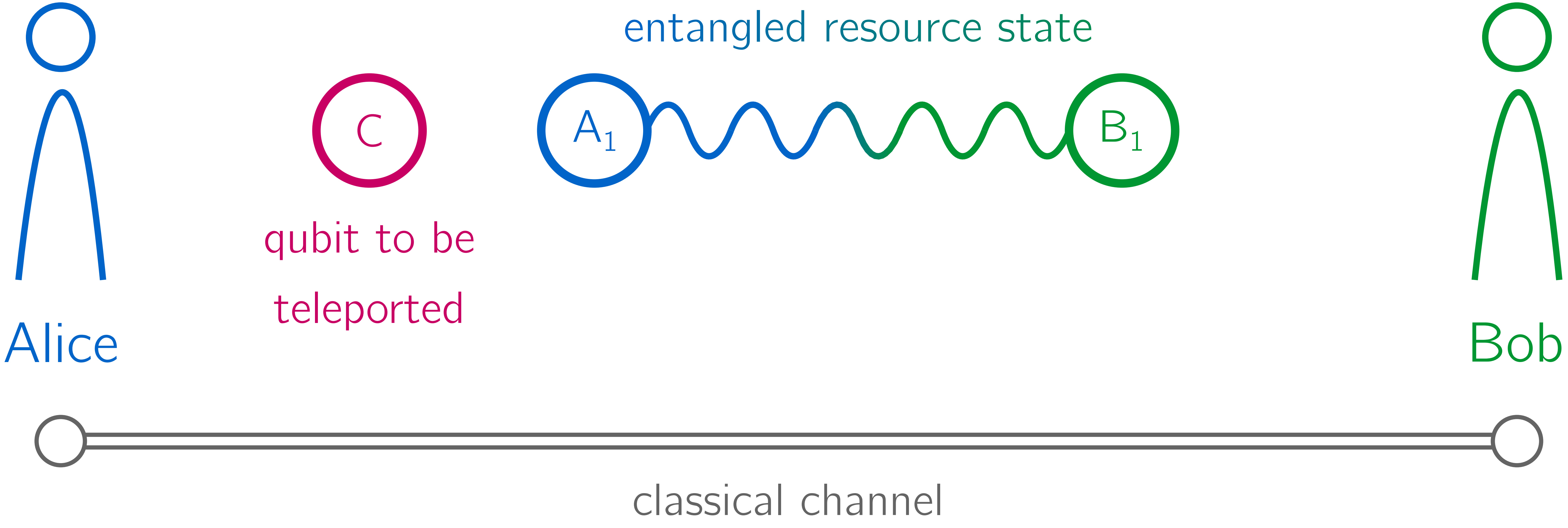
Table of contents

Standard teleportation protocol

Idea of teleportation:

[Bennett et al. '93]

entanglement + classical channel = quantum channel

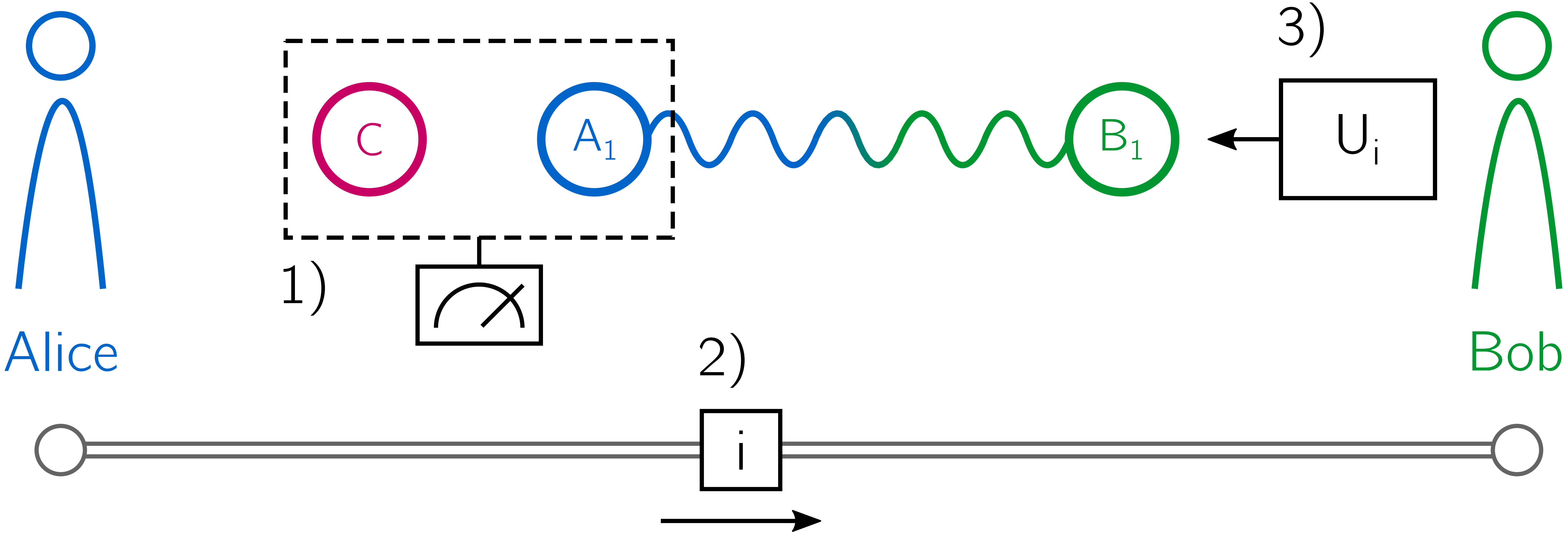


Standard teleportation protocol

Steps:

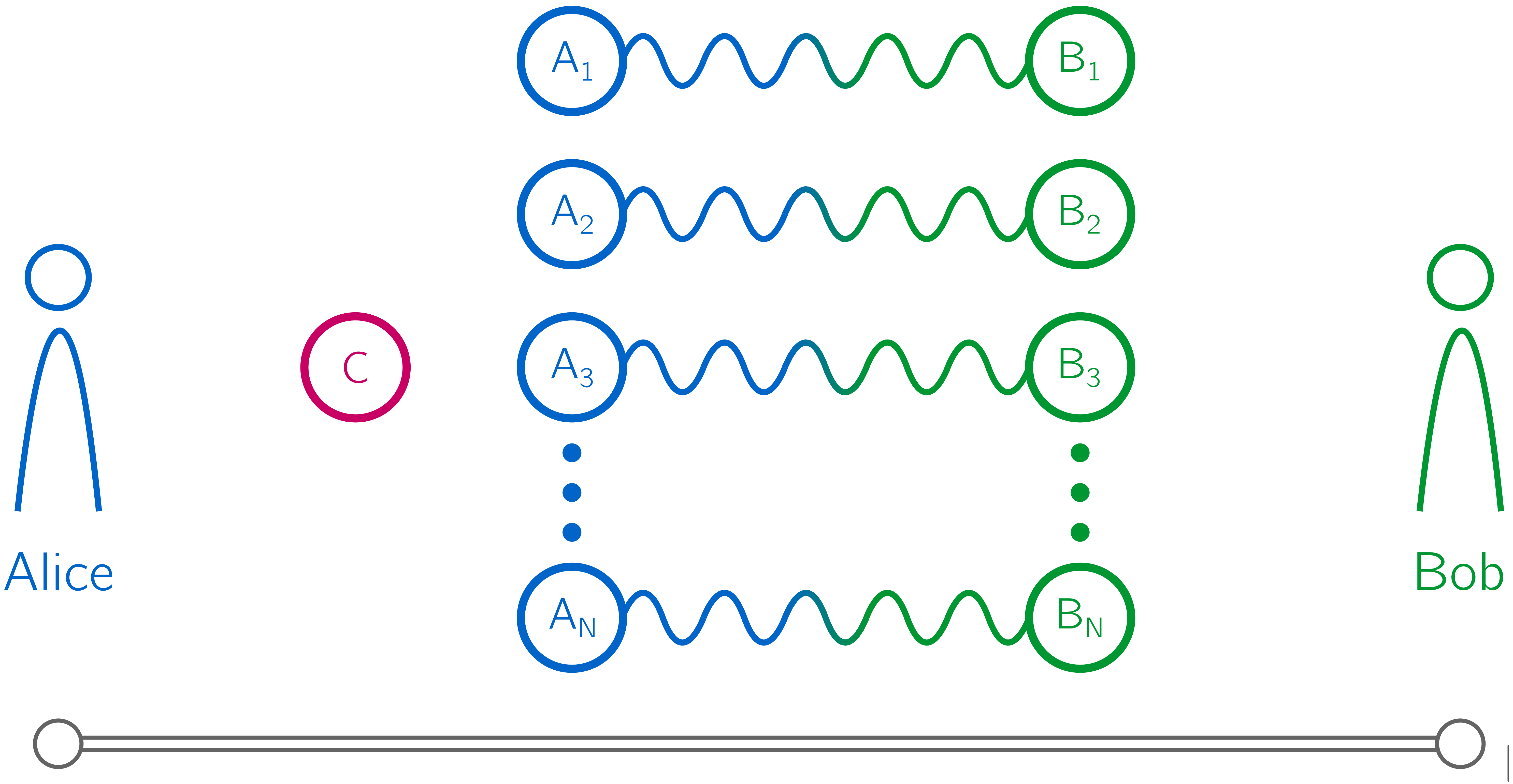
[Bennett et al. '93]

1) Alice measures CA_1.
2) Alice sends classical outcome i to Bob.
3) Bob applies correction operation U_i to B.



Port-based teleportation

[Ishizaka, Hiroshima '08]

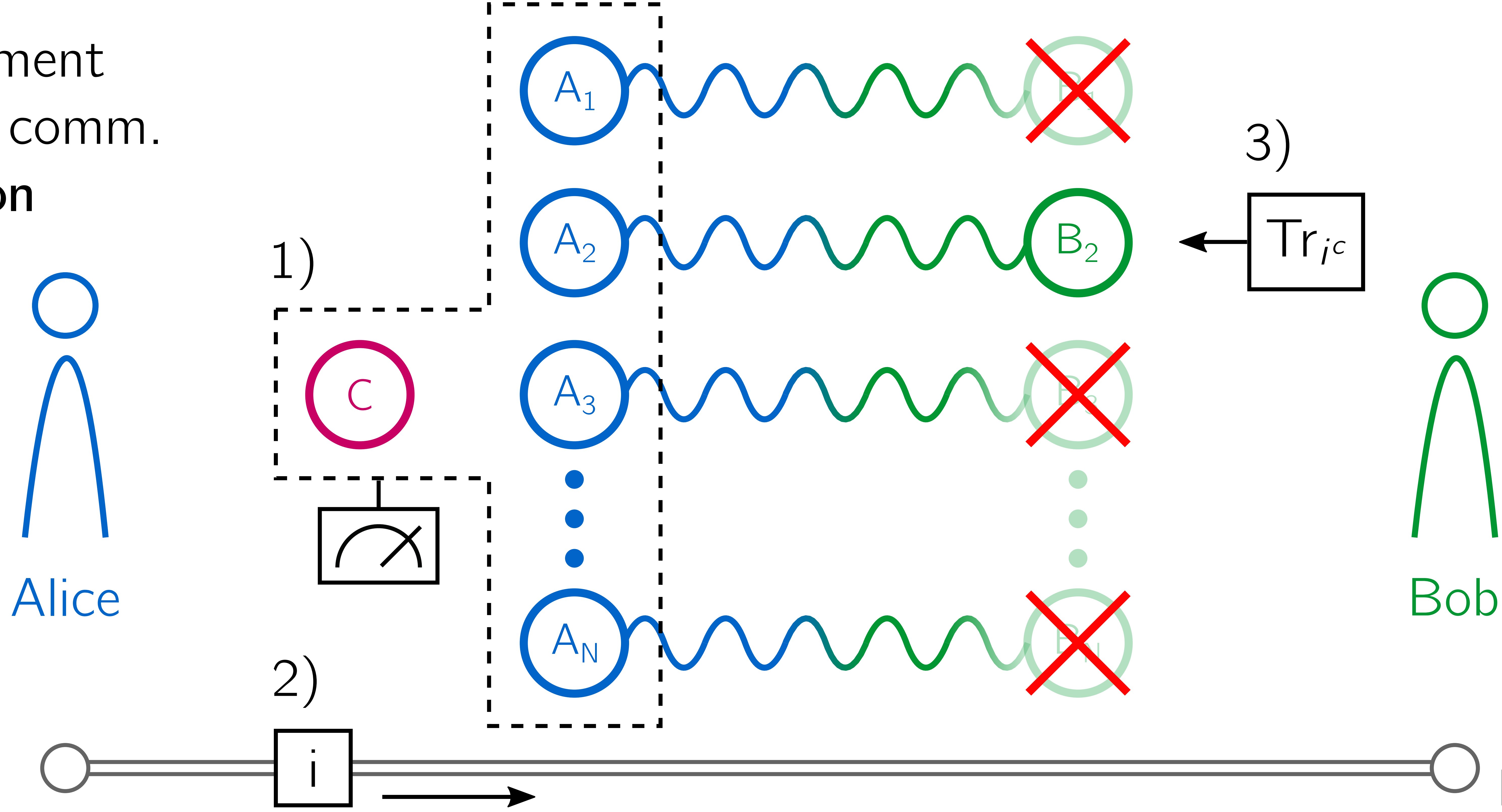


Port-based teleportation

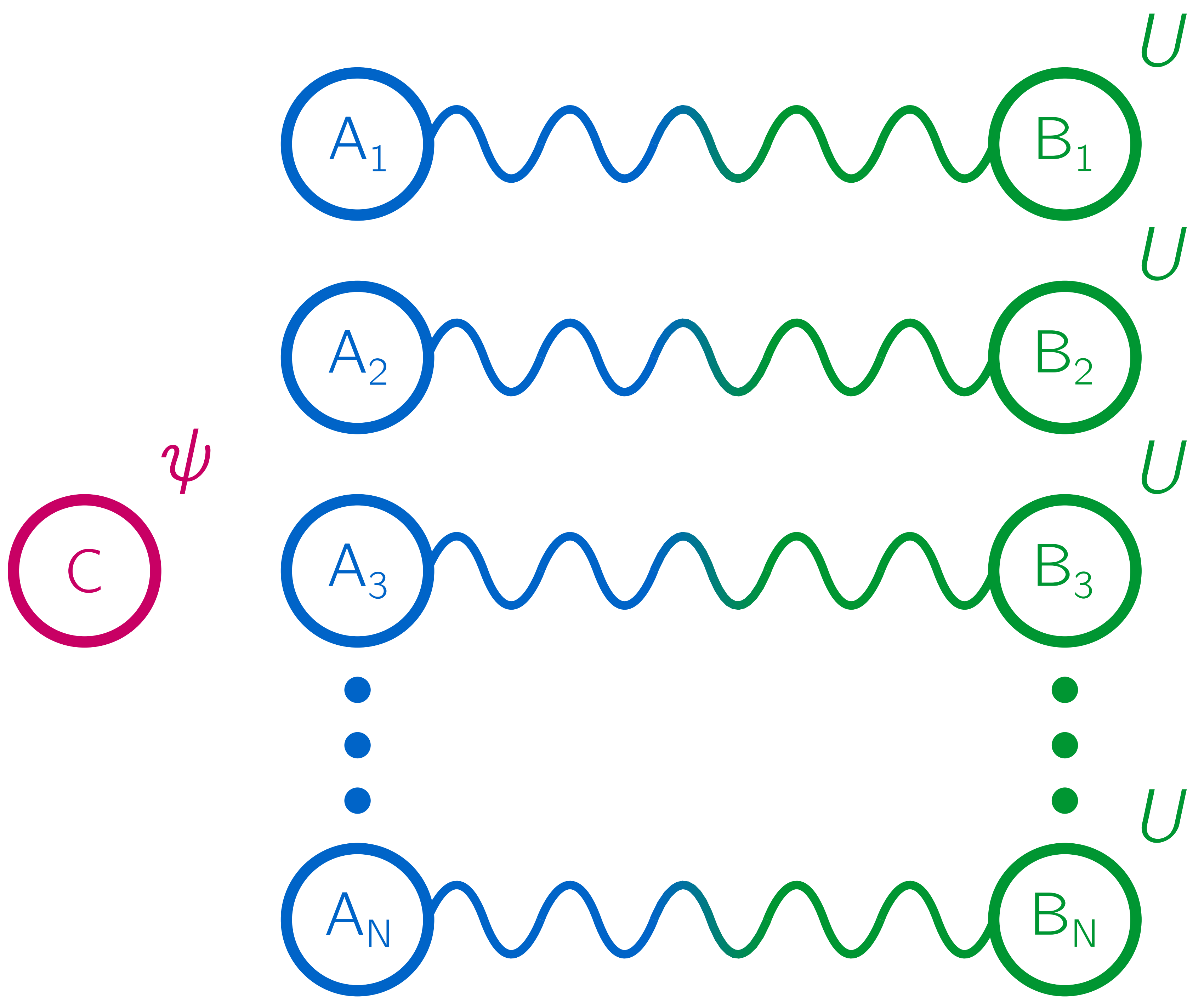
[Ishizaka, Hiroshima '08]

Steps:

1) Measurement
2) Classical comm.
3) **Correction**



Port-based teleportation



Unitary covariance:

"Correction" (partial trace) commutes with any unitary applied to all of Bob's ports.

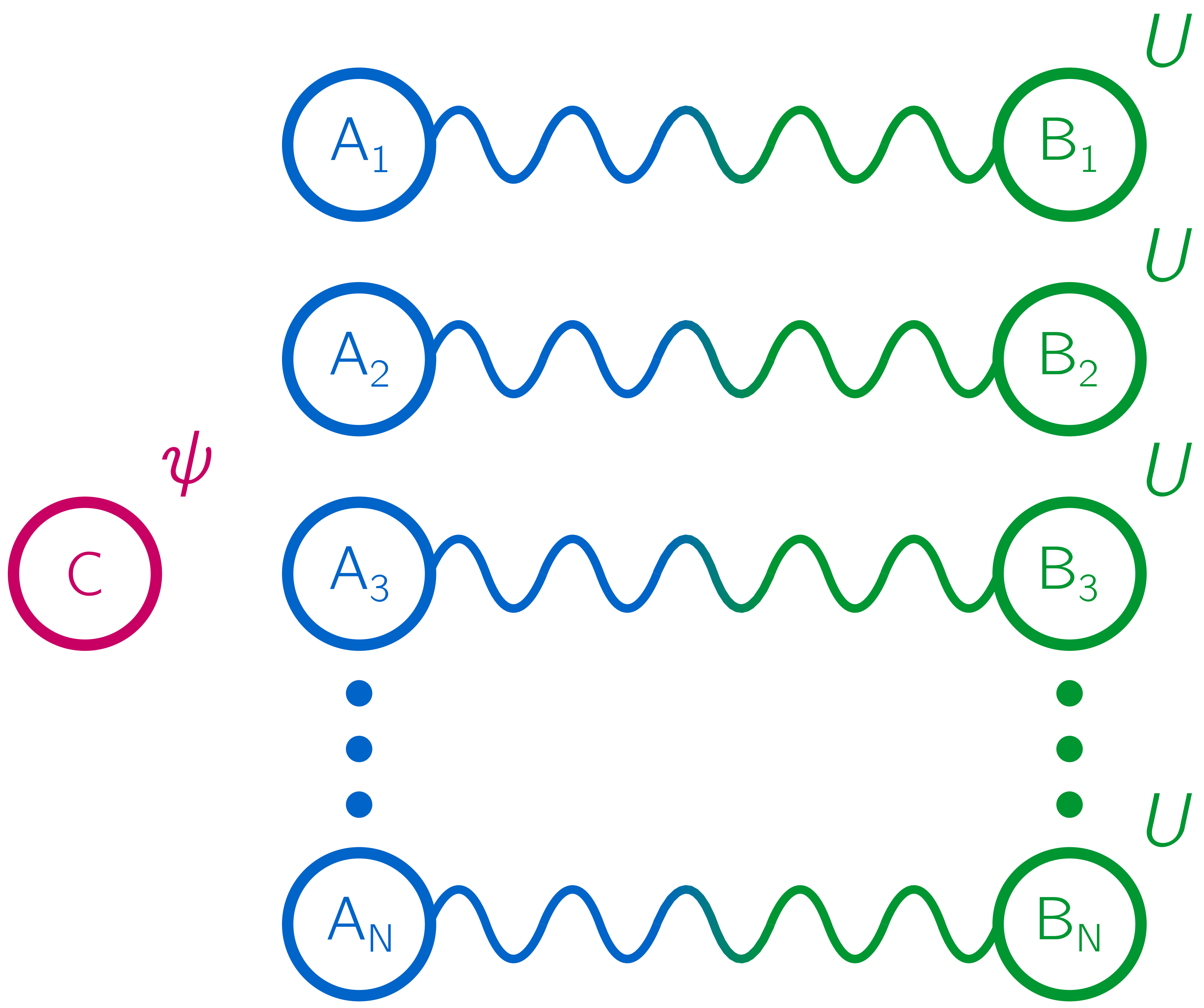
Initial state $|\psi\rangle_C$ is teleported to $U|\psi\rangle_C$.

Caveat:

Protocol cannot be perfect for finite resources. $(N < \infty)$

[Nielsen, Chuang '97]

Port-based teleportation



Nevertheless, unitary covariance enables following **applications of PBT**:

→ Universal programmable quantum processors

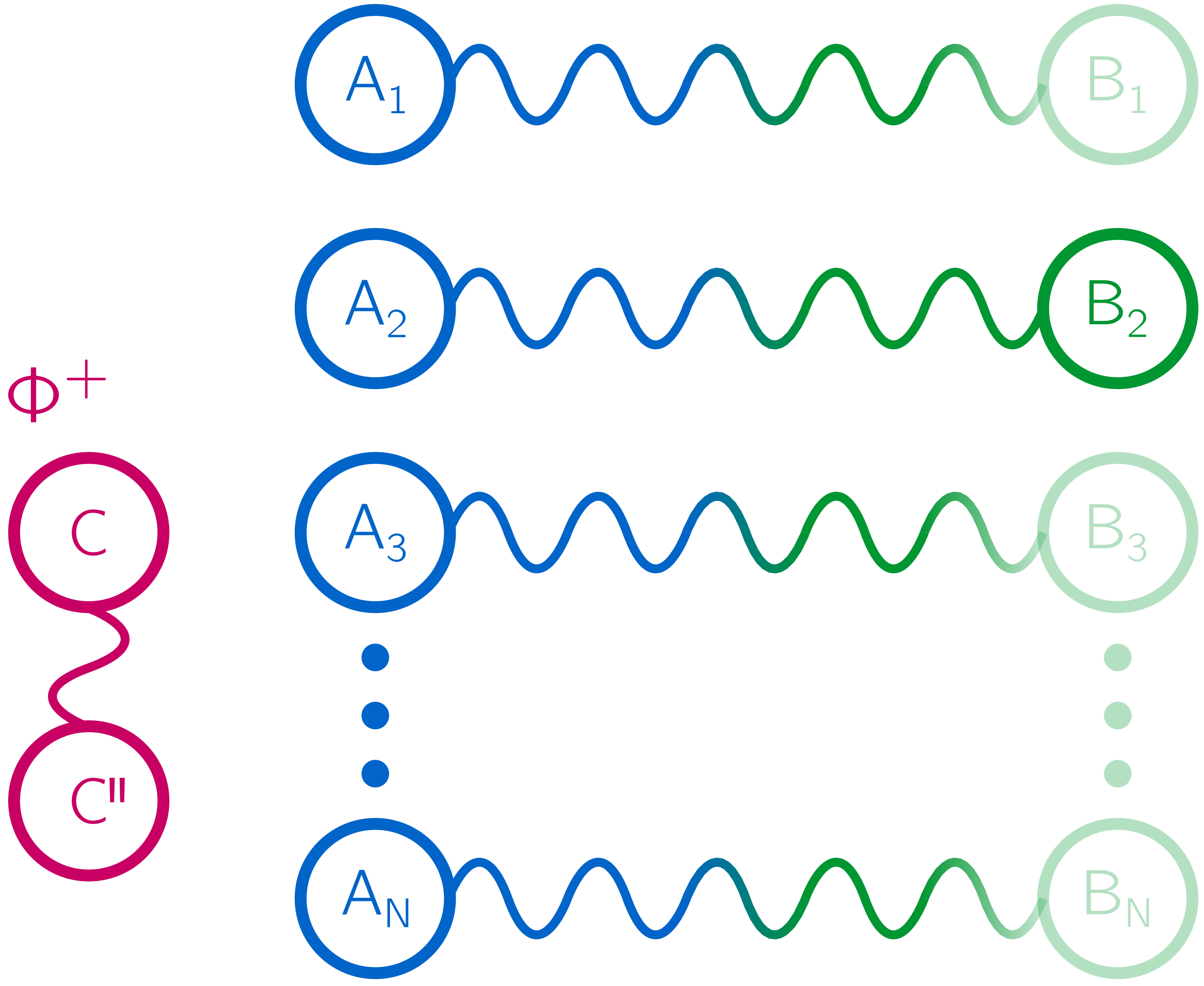
→ Attacks on position-based cryptography

→ Quantum channel discrimination

→ Entanglement-assisted quantum error correction?

[Ishizaka, Hiroshima '08; Beigi, König '11; Buhrman et al. '14; Pirandola et al. '19]

Quantifying performance of PBT



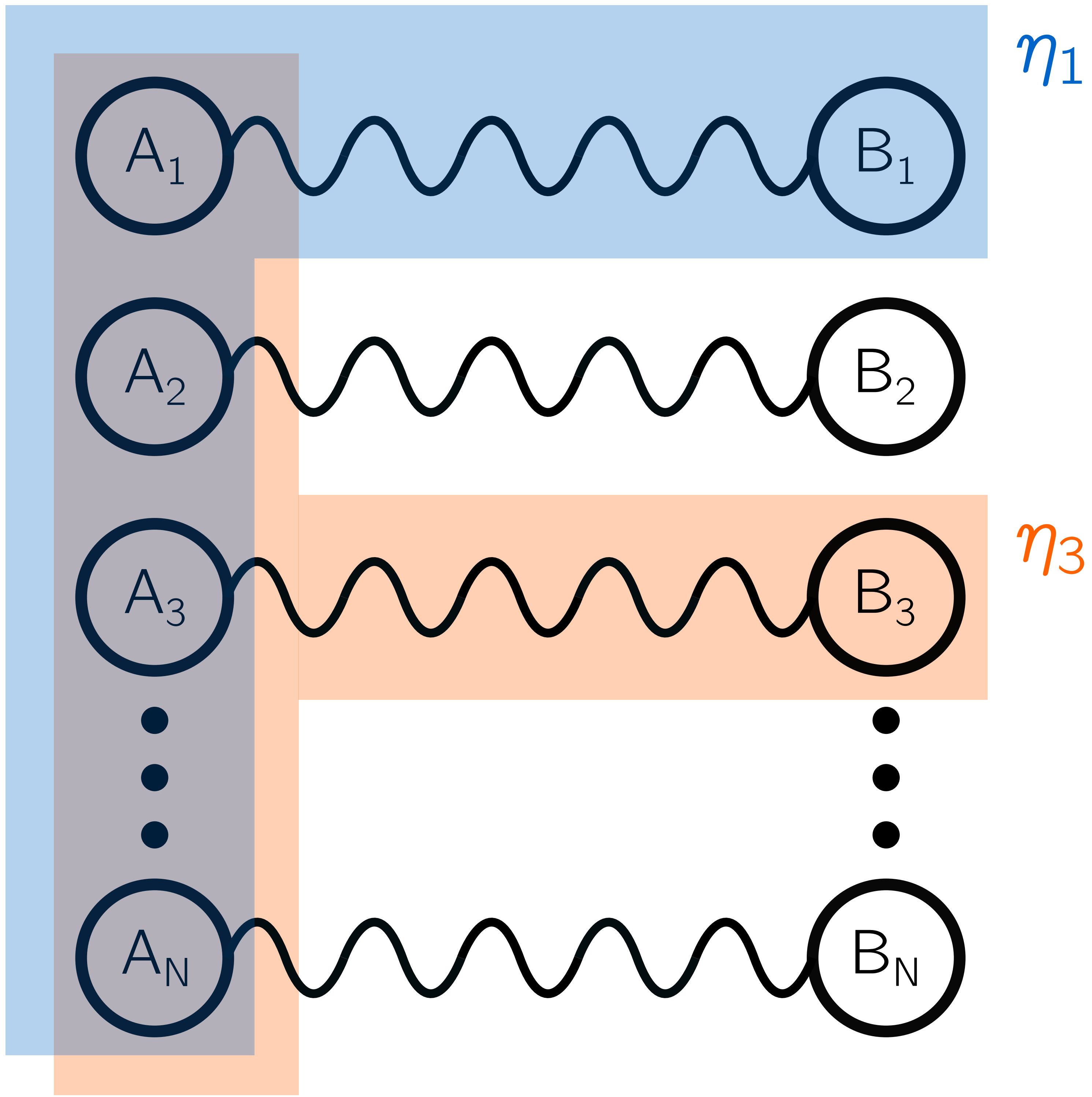
Goal of PBT:
Approximate identity channel
$C \to B_i \equiv C'$.

Let $\Lambda : C \to C'$ denote the effective **teleportation channel**.

Entanglement fidelity:

$$F(\Lambda) = \mathrm{Tr}\left[\Phi^+_{C'C''}\,(\Lambda \otimes \mathrm{id})\,(\Phi^+_{CC''})\right]$$

PBT and state discrimination



Fundamental insight:

Teleporting C of $\Phi^+_{CC''}$ through ports is equivalent to distinguishing states $\eta_i \equiv \eta_{A^N B_i}$ with uniform prior $\frac{1}{N}$:

$$F(\Lambda) = \frac{N}{d^2} p_{\text{succ}}$$

Equivalence holds more generally for **arbitrary port states** $\rho_{A^N B^N}$.

[Ishizaka, Hiroshima '08]

Semidefinite programming

State discrimination problem: distinguish states η_i with prior probabilities p_i.

Primal problem P

$$\text{Maximize: } \sum_{i=1}^{N} p_i \operatorname{Tr}(\eta_i E_i)$$

$$\text{subject to: } E_i \geq 0 \text{ for all } i,$$

$$\sum_{i=1}^{N} E_i = \mathbb{1}.$$

Dual problem D

$$\text{Minimize: } \operatorname{Tr} K$$

$$\text{subject to: } K \geq p_i \eta_i \text{ for all } i.$$

Strong duality: $p_{\text{succ}} = P = D$.

Semidefinite programming

State discrimination problem: distinguish states η_i with prior probabilities p_i.

Primal problem P

$$\text{Maximize: } \sum_{i=1}^{N} p_i \operatorname{Tr}(\eta_i E_i)$$

$$\text{subject to: } E_i \geq 0 \text{ for all } i, \quad \sum_{i=1}^{N} E_i = \mathbb{1}.$$

Dual problem D

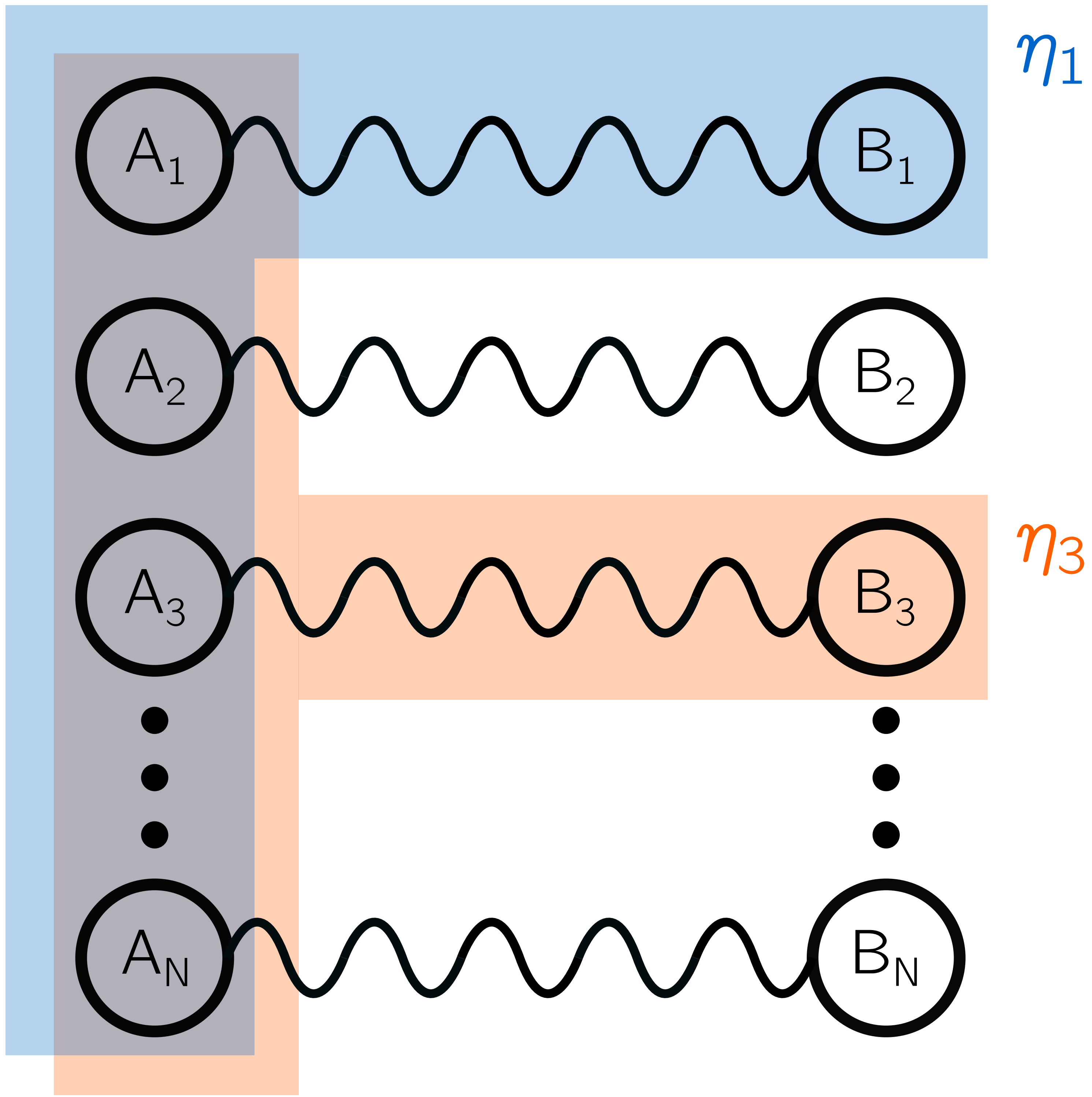
$$\text{Minimize: } \operatorname{Tr} K$$

$$\text{subject to: } K \geq p_i \eta_i \text{ for all } i.$$

POVM: most general definition of quantum measurement

Strong duality: $p_{\text{succ}} = P = D$.

PBT and state discrimination



Port state: N max. entangled states

$$\rho_{A^N B^N} = (\Phi^+_{AB})^{\otimes N}$$

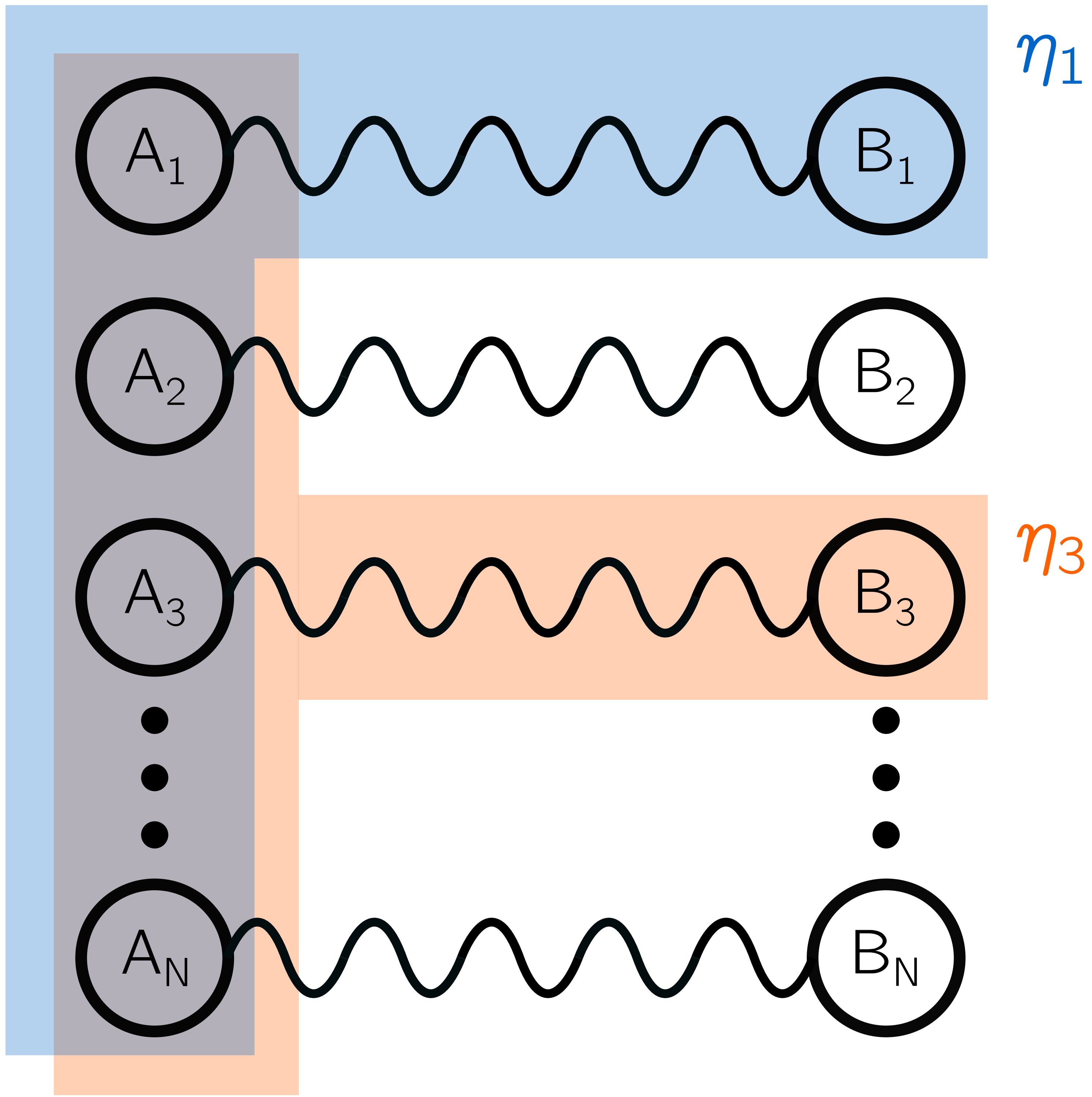
State discrimination problem:

$$\eta_i = \Phi^+_{A_i B_i} \otimes \left(\tfrac{1}{d}\mathbb{1}_A\right)^{\otimes N-1}$$

$$p_i = \frac{1}{N}$$

What is a good choice for the POVM (measurement)?

Pretty good measurement



Define average state $\bar{\eta} = \sum_{i=1}^{N} p_i \eta_i$.

Measurement operators:

$$E_i = \bar{\eta}^{-1/2}\, p_i \eta_i\, \bar{\eta}^{-1/2}$$

Also called square root measurement.

Easy to check:

1) $E_i \geq 0$ for all i;

2) $\sum_i E_i = \operatorname{supp} \bar{\eta}$.

[Hausladen, Wootters '94]

Table of contents

→ Quantum information 101 & 102

→ Port-based teleportation: Definition and properties

→ **Symmetries of port-based teleportation**

→ Asymptotics of port-based teleportation

→ Generalizations and concluding remarks

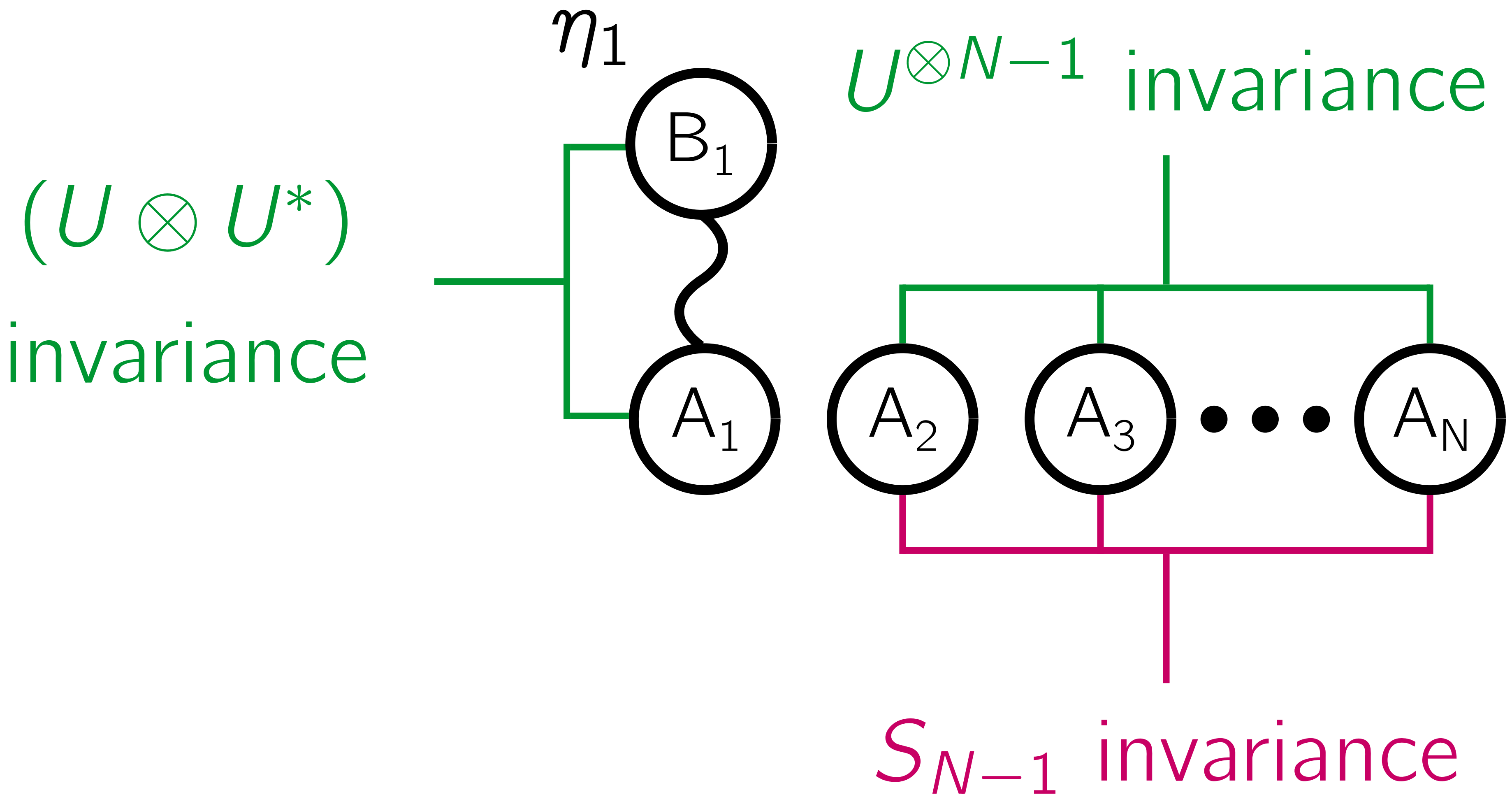
Symmetries in state discrimination problem

Fundamental symmetry:

$$(U \otimes U^*)|\Phi^+\rangle_{AB} = |\Phi^+\rangle_{AB} \text{ for all unitaries } U.$$

State ensemble:

$$\eta_i = \Phi^+_{A_i B_i} \otimes \left(\tfrac{1}{d}\mathbb{1}_A\right)^{\otimes N-1}$$



Resulting symmetries:

$$\left[U^{\otimes N} \otimes U^*,\, \eta_i\right] = 0$$

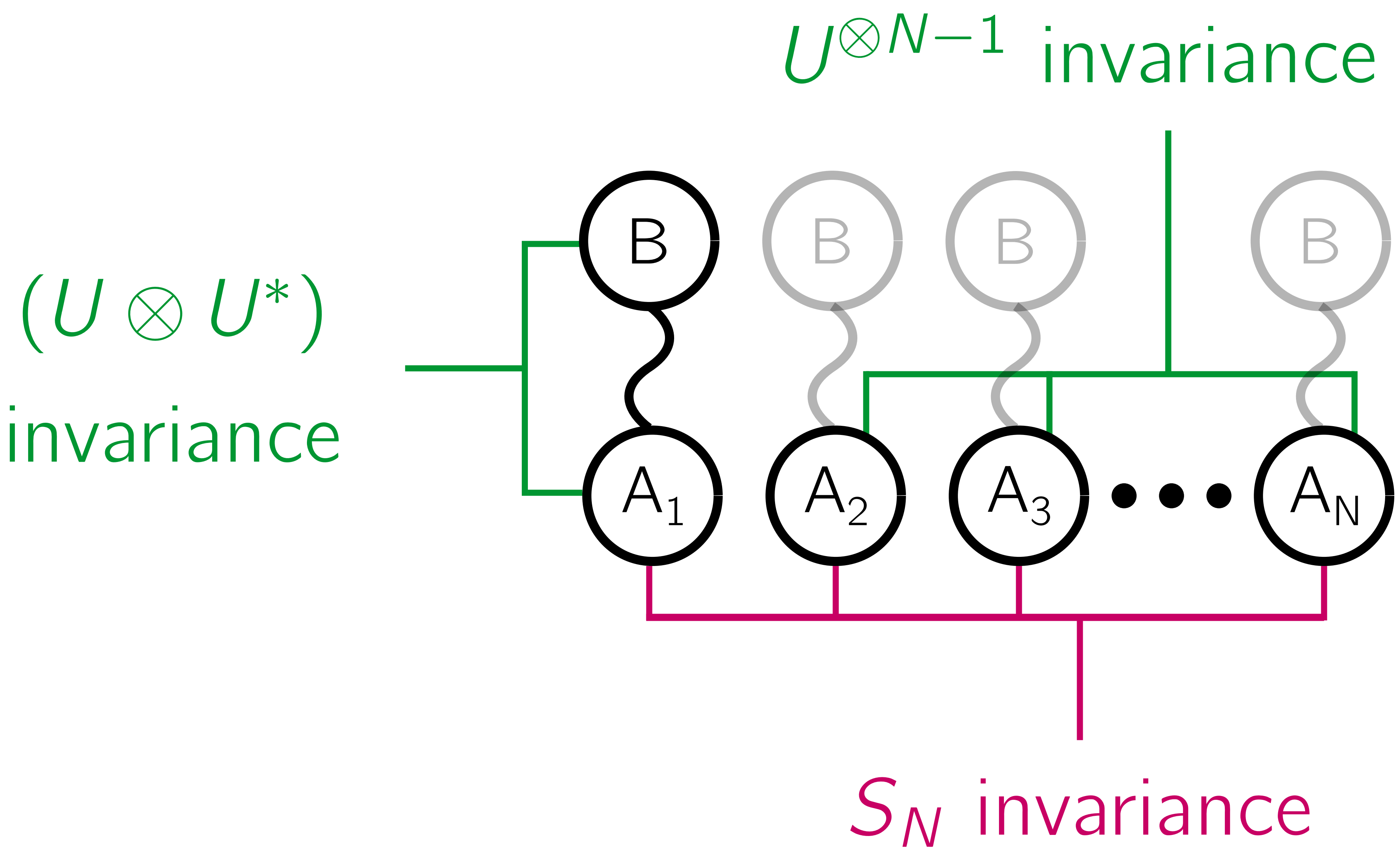
$$[\mathbb{1}_{A_i B_i} \otimes \pi,\, \eta_i] = 0 \qquad (\pi \in S_{N-1})$$

Symmetries in state discrimination problem

Average ensemble state:

$$\bar{\eta} = \Phi^+_{A_1 B_1} \otimes (\tfrac{1}{d}\mathbb{1}_A)^{\otimes N-1} + \cdots + \Phi^+_{A_N B_N} \otimes (\tfrac{1}{d}\mathbb{1}_A)^{\otimes N-1}$$

$$(B_i \equiv B)$$



Resulting symmetries:

$$\left[U^{\otimes N} \otimes U^*, \, \bar{\eta}\right] = 0$$

$$[\mathbb{1}_B \otimes \pi, \, \bar{\eta}] = 0 \quad (\pi \in S_N)$$

Symmetries on tensor product spaces

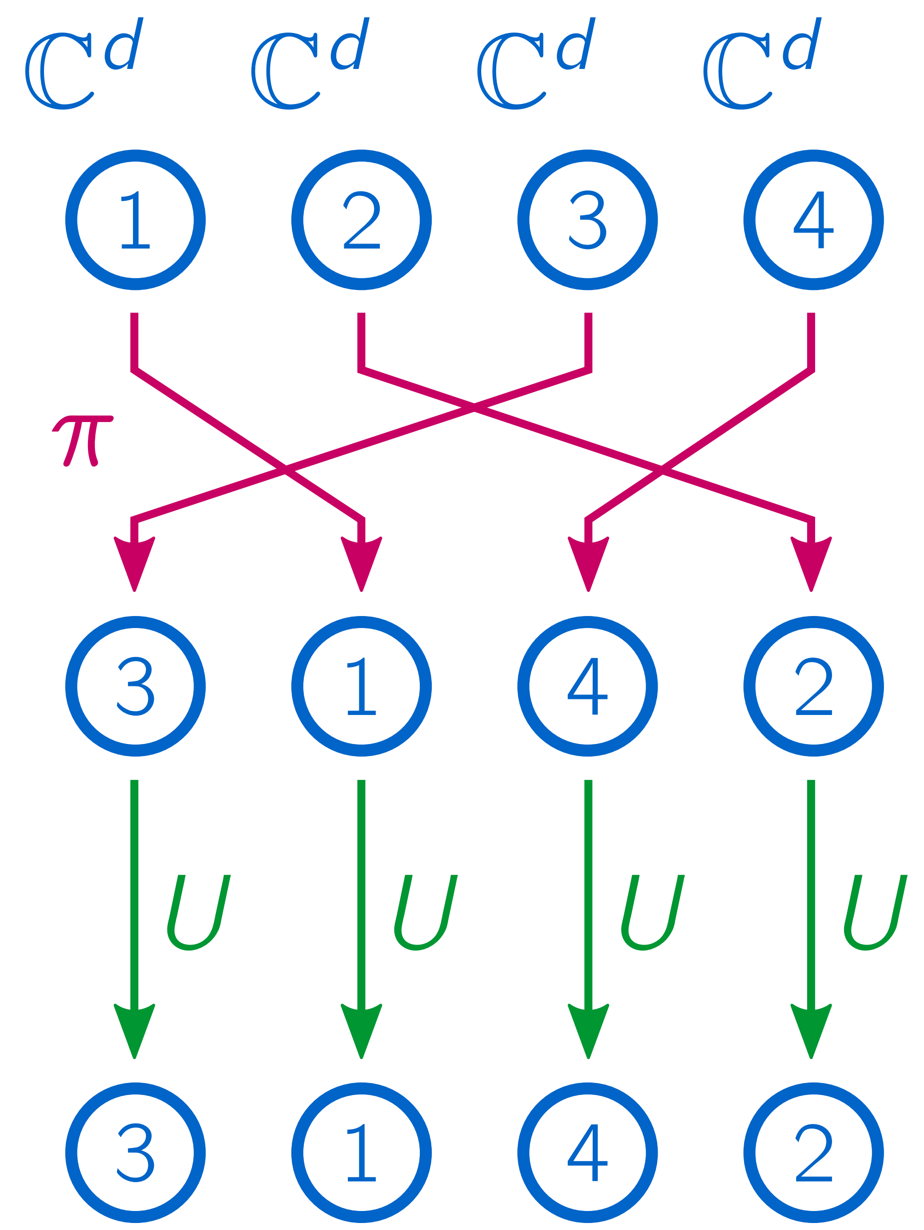
Representation space $(\mathbb{C}^d)^{\otimes N}$.

Symmetric group S_N:

$$S_N \ni \pi : |\psi_1\rangle \otimes \ldots \otimes |\psi_N\rangle \longmapsto |\psi_{\pi^{-1}(1)}\rangle \otimes \ldots \otimes |\psi_{\pi^{-1}(N)}\rangle$$

Unitary group $\mathcal{U}_d$:

$$\mathcal{U}_d \ni U : |\psi_1\rangle \otimes \ldots \otimes |\psi_N\rangle \longmapsto U|\psi_1\rangle \otimes \ldots \otimes U|\psi_N\rangle$$



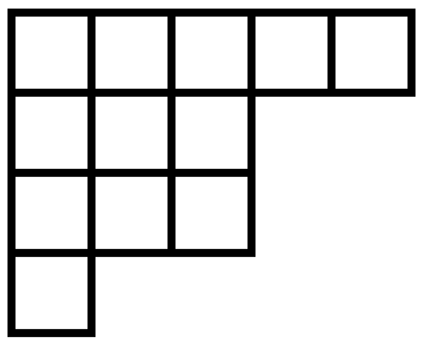
These two representations commute and span each other's commutant.

$\longrightarrow$ **Schur-Weyl duality** decomposes $(\mathbb{C}^d)^{\otimes N}$ "nicely" into S_N and $\mathcal{U}_d$ irreps.

Irreducible representations

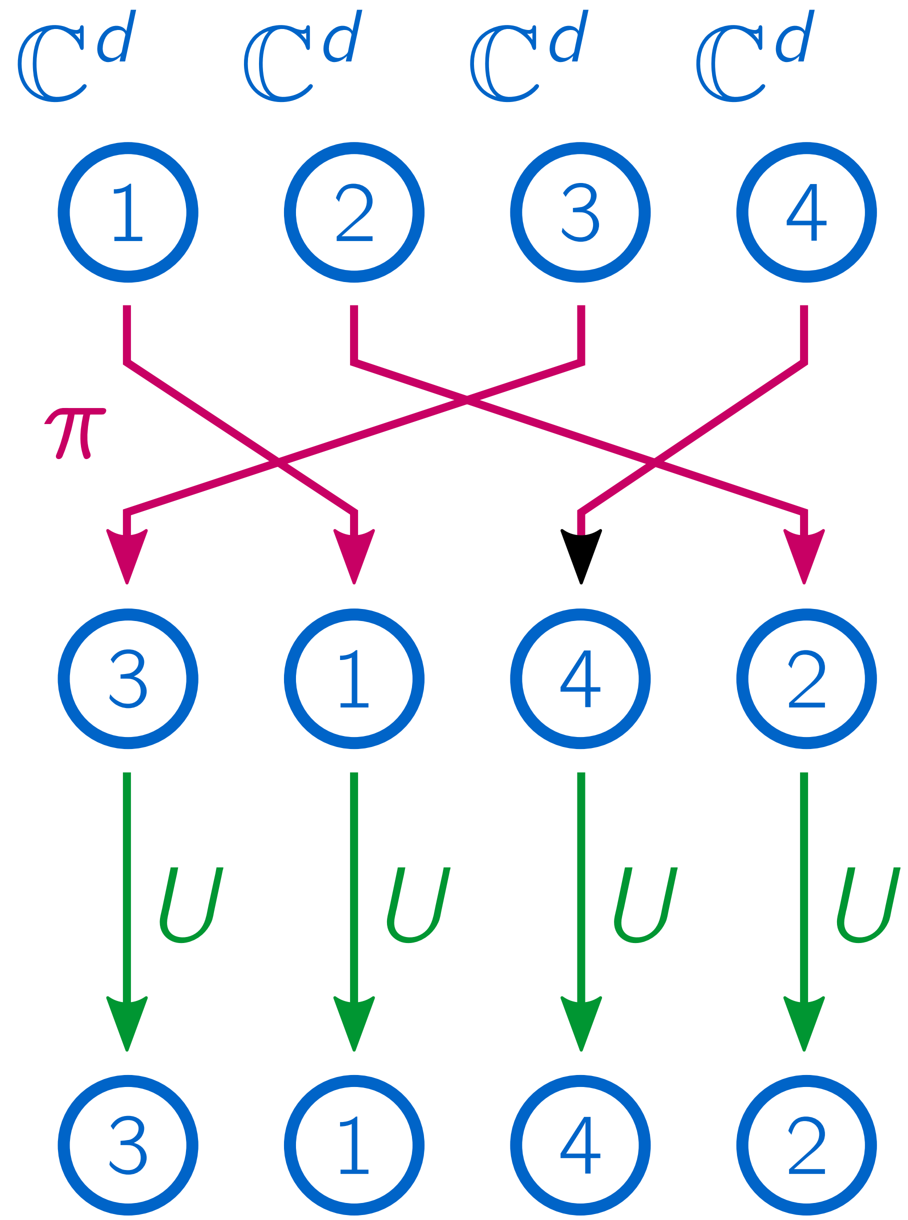
Partitions $\mu = (\mu_1, \ldots, \mu_d) \vdash_d N$

$\longleftrightarrow$ Young diagrams



$N = 12, d = 4,$

$\mu = (5, 3, 3, 1)$



Irreps of S_N: **Specht modules** W_μ

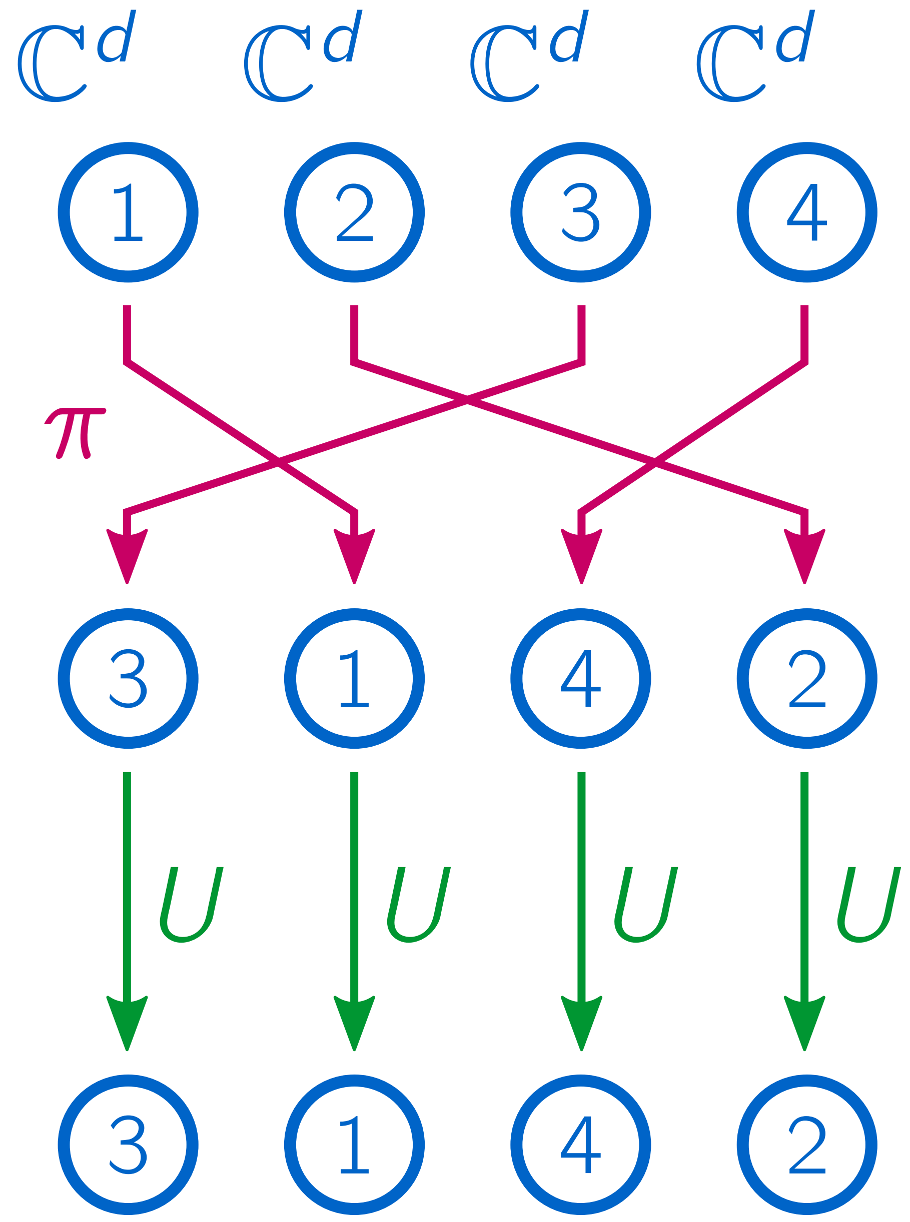
Dimension: $d_\mu := \dim W_\mu$

Irreps of $\mathcal{U}_d$: **Weyl modules** V_μ^d

Dimension: $m_{d,\mu} := \dim V_\mu^d$

Schur-Weyl duality

S_N and $\mathcal{U}_d$ span each other's commutants on $(\mathbb{C}^d)^{\otimes N}$.



Schur-Weyl decomposition:

$$(\mathbb{C}^d)^{\otimes N} = \bigoplus_{\mu \vdash_d N} V^d_\mu \otimes W_\mu$$

Application of **Schur's Lemma**:

If state ρ on $(\mathbb{C}^d)^{\otimes N}$ is invariant under S_N and $\mathcal{U}_d$:

$$\rho = \bigoplus_{\mu \vdash_d N} r_\mu \, \mathbb{1}_{V^d_\mu} \otimes \mathbb{1}_{W_\mu}, \qquad \text{where } r_\mu \geq 0 \text{ and } \sum_{\mu \vdash_d N} r_\mu m_{d,\mu} d_\mu = 1.$$

Pieri rule

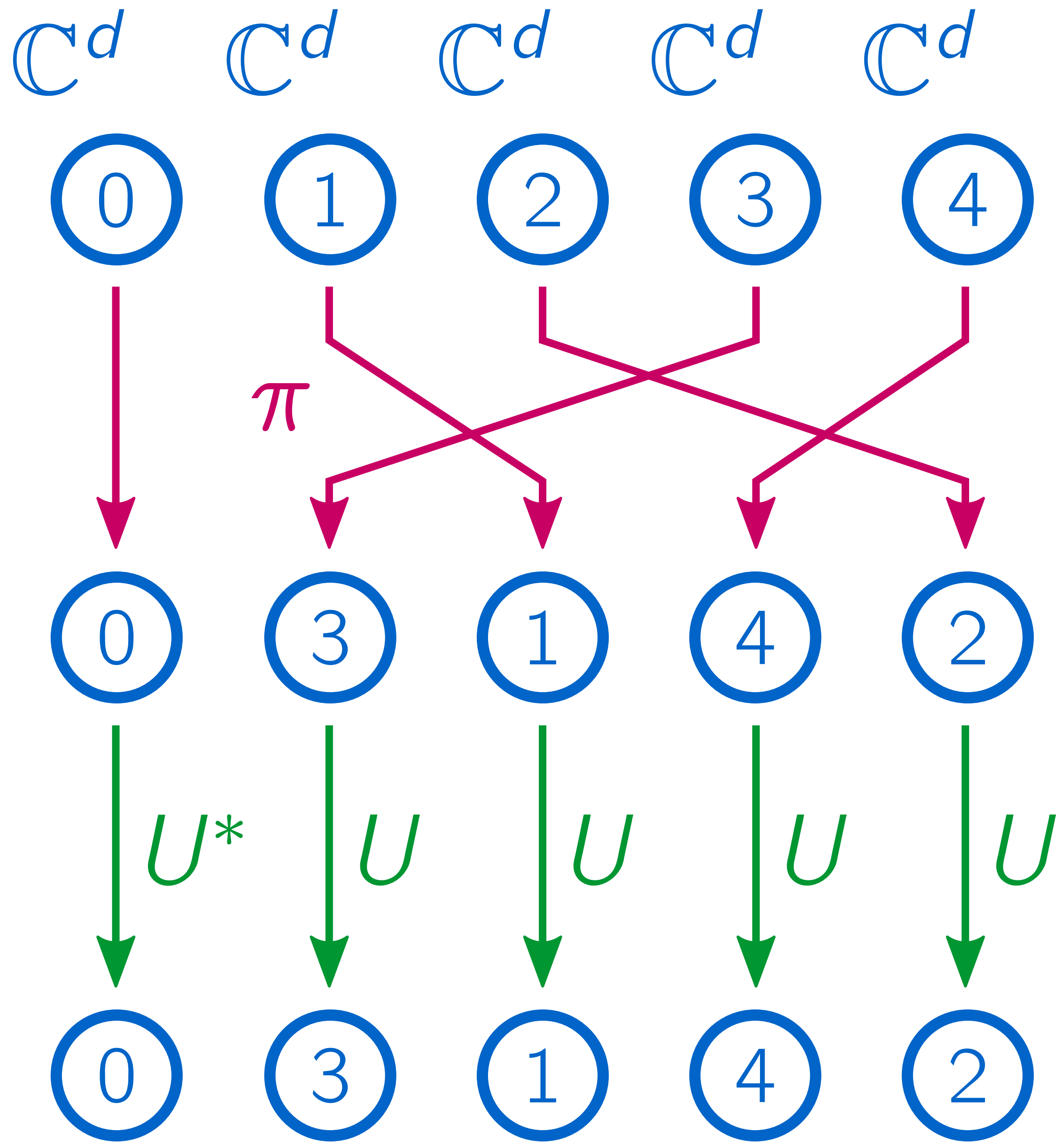
Ensemble states η_i and average state $\bar{\eta}$ have $U^* \otimes U^{\otimes N}$ symmetry.

Incorporate $U^* \otimes U^{\otimes N}$ symmetry into Schur-Weyl decomposition using **Pieri rule**:

$$(\mathbb{C}^d)^* \otimes V^d_\mu = \bigoplus_{i:\, \mu_i > \mu_{i+1}} V^d_{\mu - \varepsilon_i}$$

Resulting block-diagonal form of $\bar{\eta}$:

$$\bar{\eta} = \bigoplus_{\mu \vdash_d N} \bigoplus_{\alpha = \mu - \square} s_{\mu,\alpha} \, \mathbb{1}_{V^d_\alpha} \otimes \mathbb{1}_{W_\mu}$$



$$(\mathbb{C}^d)^{\otimes N+1} = (\mathbb{C}^d)^* \otimes \bigoplus_{\mu \vdash_d N} V^d_\mu \otimes W_\mu$$

Solving the state discrimination problem

Discriminate states $\eta_i = \Phi^+_{A_i B_i} \otimes (\frac{1}{d}\mathbb{1}_A)^{\otimes N-1}$ (with uniform prior).

Pretty good measurement: $E_i = \bar{\eta}^{-1/2}\,\eta_i\,\bar{\eta}^{-1/2}$ with average state $\bar{\eta} = \sum_i \eta_i$.

Success probability: $p_{\text{succ}} = \sum_{i=1}^{N} p_i \operatorname{Tr}(E_i \eta_i)$

$$= \frac{1}{N}\sum_{i=1}^{N} \operatorname{Tr}\left(\bar{\eta}^{-1/2}\,\eta_i\,\bar{\eta}^{-1/2}\,\eta_i\right)$$

$$= \frac{1}{Nd^N}\sum_{\alpha \vdash_d N-1}\left(\sum_{\mu=\alpha+\square}\sqrt{m_{d,\mu} d_\mu}\right)^2$$

Use symmetries from Schur-Weyl and Pieri rule!

[Studzinski et al. '17]

Optimality of pretty good measurement

Success probability: $p_{\mathrm{succ}} = \frac{1}{Nd^N} \sum_{\alpha \vdash_d N-1} \Big(\sum_{\mu = \alpha + \square} \sqrt{m_{d,\mu} d_\mu} \Big)^2$

Optimality of PGM via SDP duality: $p_{\mathrm{succ}} = \min\{\operatorname{Tr} K : K \geq p_i \eta_i \text{ for all } i\}$.

Show that $K = \frac{1}{N} \sum_{i=1}^{N} \bar{\eta}^{-1/4}\, \eta_i\, \bar{\eta}^{-1/2}\, \eta_i\, \bar{\eta}^{-1/4}$ is dual feasible:

$$\sum_{i=1}^{N} \bar{\eta}^{-1/4}\, \eta_i\, \bar{\eta}^{-1/2}\, \eta_i\, \bar{\eta}^{-1/4} \geq \eta_i \text{ for all } i.$$

For this choice:

$\operatorname{Tr} K = p_{\mathrm{succ}}$.

Table of contents

Asymptotics of PBT performance

Entanglement fidelity: $F(\Lambda) = \frac{N}{d^2} p_{\text{succ}}$

$$= \frac{1}{d^{N+2}} \sum_{\alpha \vdash_d N-1} \Big(\sum_{\mu = \alpha + \square} \sqrt{m_{d,\mu} d_\mu} \Big)^2$$

Good: Beautiful closed formula in terms of representation-theoretic data.

Bad: Hard to tell what happens for large number of ports and fixed local dimension.

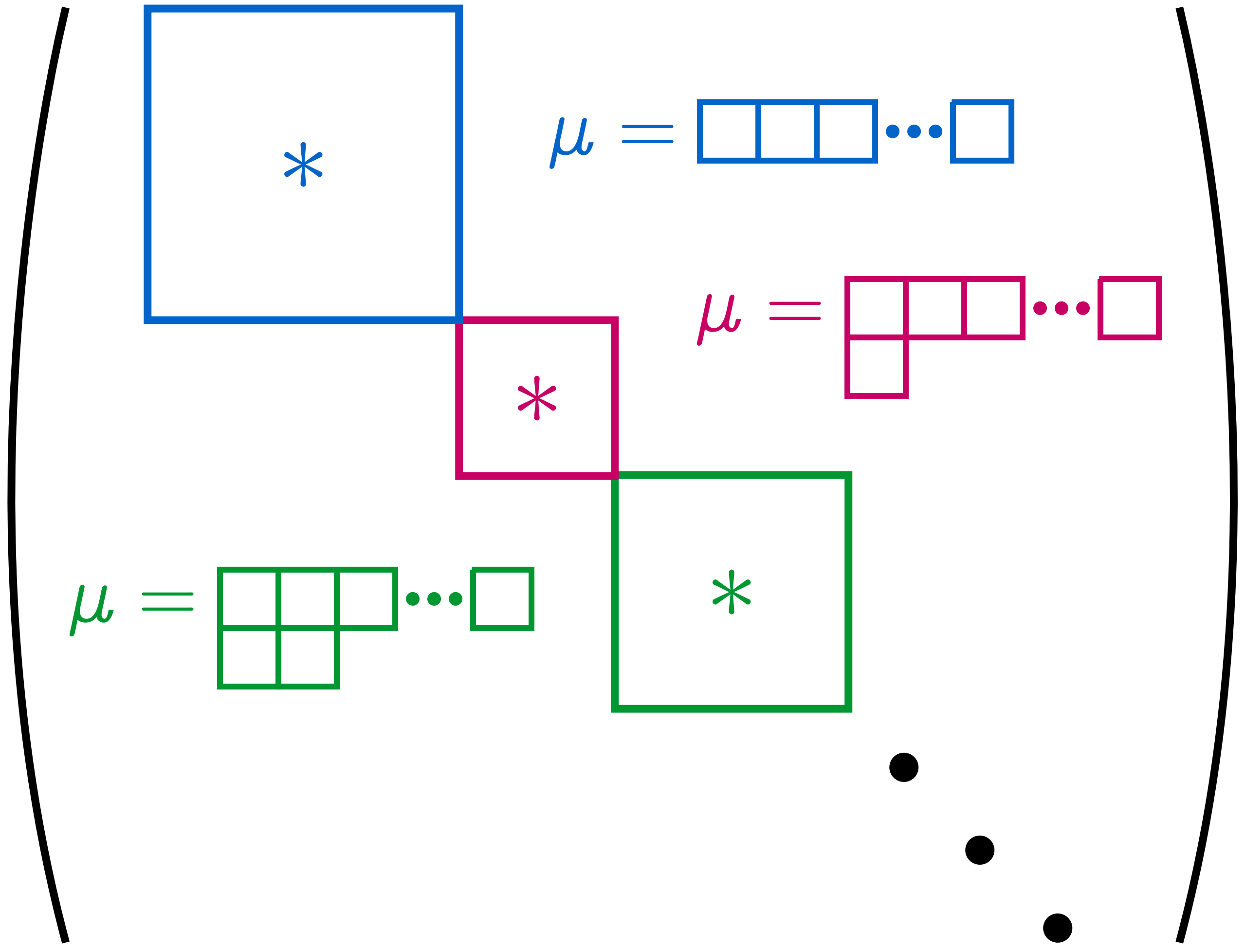
Asymptotic limit: $d \geq 2$ fixed but arbitrary, $N \to \infty$

[Studzinski et al. '17]

Schur-Weyl distribution and spectrum estimation

Recall Schur-Weyl duality: $(\mathbb{C}^d)^{\otimes N} = \bigoplus_{\mu \vdash_d N} V_\mu^d \otimes W_\mu$

Denote by P_μ the projection onto $V_\mu^d \otimes W_\mu$.



Schur-Weyl distribution: $p_{d,N}(\mu) = \frac{1}{d^N} \operatorname{Tr} P_\mu$

$$= \frac{m_{d,\mu} d_\mu}{d^N}$$

Spectrum estimation:

Let $\mathbf{X} \sim p_{d,N}(\mu)$, then

$$\frac{1}{N}\mathbf{X} \xrightarrow{N\to\infty} (\tfrac{1}{d}, \ldots, \tfrac{1}{d}) \text{ in distribution.}$$

[Alicki '88], [Keyl, Werner '01]

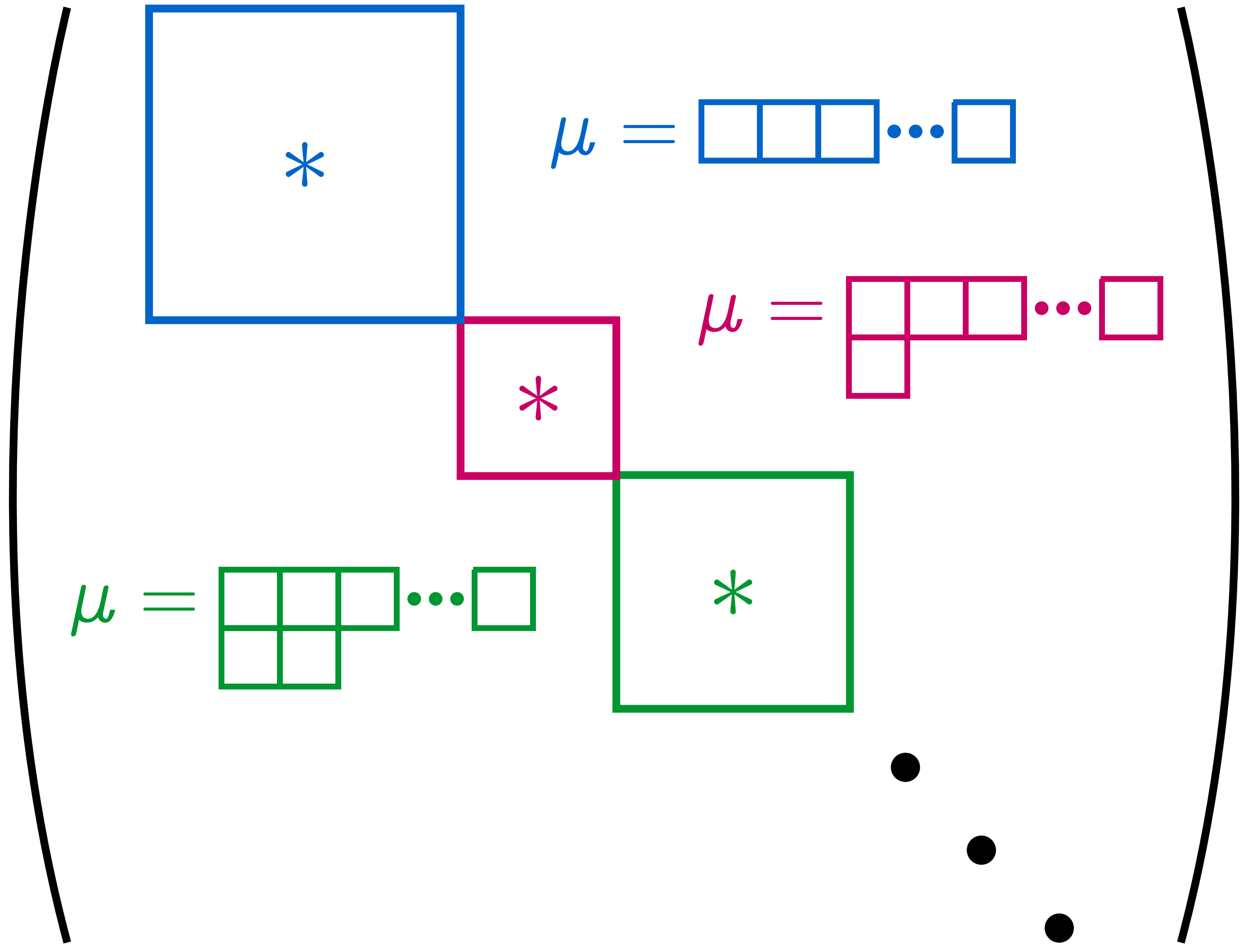
Fluctuations of the Schur-Weyl distribution

Spectrum estimation:

Let $\mathbf{X} \sim p_{d,N}(\mu)$, then $\frac{1}{N}\mathbf{X} \xrightarrow{N\to\infty} (\frac{1}{d}, \ldots, \frac{1}{d})$ in distribution.

Center and normalize YD's:

$$\mathbf{Y} = \sqrt{\tfrac{d}{N}}\left(\mathbf{X} - (\tfrac{N}{d}, \ldots, \tfrac{N}{d})\right)$$



"Central limit theorem":

$$\mathbf{Y} \xrightarrow{N\to\infty} \mathrm{spec}(\mathbf{G}) \text{ in distribution,}$$

where $\mathbf{G} \sim \mathrm{GUE}_0(d)$ is drawn from the traceless Gaussian unitary ensemble.

[Alicki '88], [Keyl, Werner '01], [Johansson '01]

Asymptotics of PBT performance

“Central limit theorem”: $\mathbf{Y} \xrightarrow{N\to\infty}$ spec($\mathbf{G}$) in distribution.

Entanglement fidelity: $F(\Lambda) = \frac{1}{d^{N+2}} \sum_{\alpha \vdash_d N-1} \Big(\sum_{\mu=\alpha+\square} \sqrt{m_{d,\mu} d_\mu} \Big)^2$

Idea: Rewrite fidelity as expectation value

$$F(\Lambda) = \mathop{\mathbb{E}}_{\boldsymbol{\alpha} \vdash_d N-1} [f(\boldsymbol{\alpha})] \text{ for a suitable function } f$$

and use CLT above to calculate with $\tilde{f}$(spec($\mathbf{G}$)) instead (much easier!).

Asymptotics of PBT performance

"Central limit theorem": $\mathbf{Y} \xrightarrow{N\to\infty} \mathrm{spec}(\mathbf{G})$ in distribution.

Ent. fidelity: $F(\Lambda) = \frac{1}{d^{N+2}} \sum_{\alpha \vdash_d N-1} \Big(\sum_{\mu=\alpha+\square} \sqrt{m_{d,\mu} d_\mu} \Big)^2 = \mathop{\mathbb{E}}_{\boldsymbol{\alpha} \vdash_d N-1} [f(\boldsymbol{\alpha})]$

Need: Stronger convergence of expectation values for suitable functions f

$\longrightarrow$ main technical result in [arXiv:1809.10751].

Main result: Asymptotic behavior of entanglement fidelity

$$F(\Lambda) = 1 - \frac{d^2-1}{4}\frac{1}{N} + O(N^{-\frac{3}{2}+\delta}) \qquad (\text{hence } F(\Lambda) \to 1 \text{ as } N \to \infty)$$

Table of contents

Fully optimized port-based teleportation

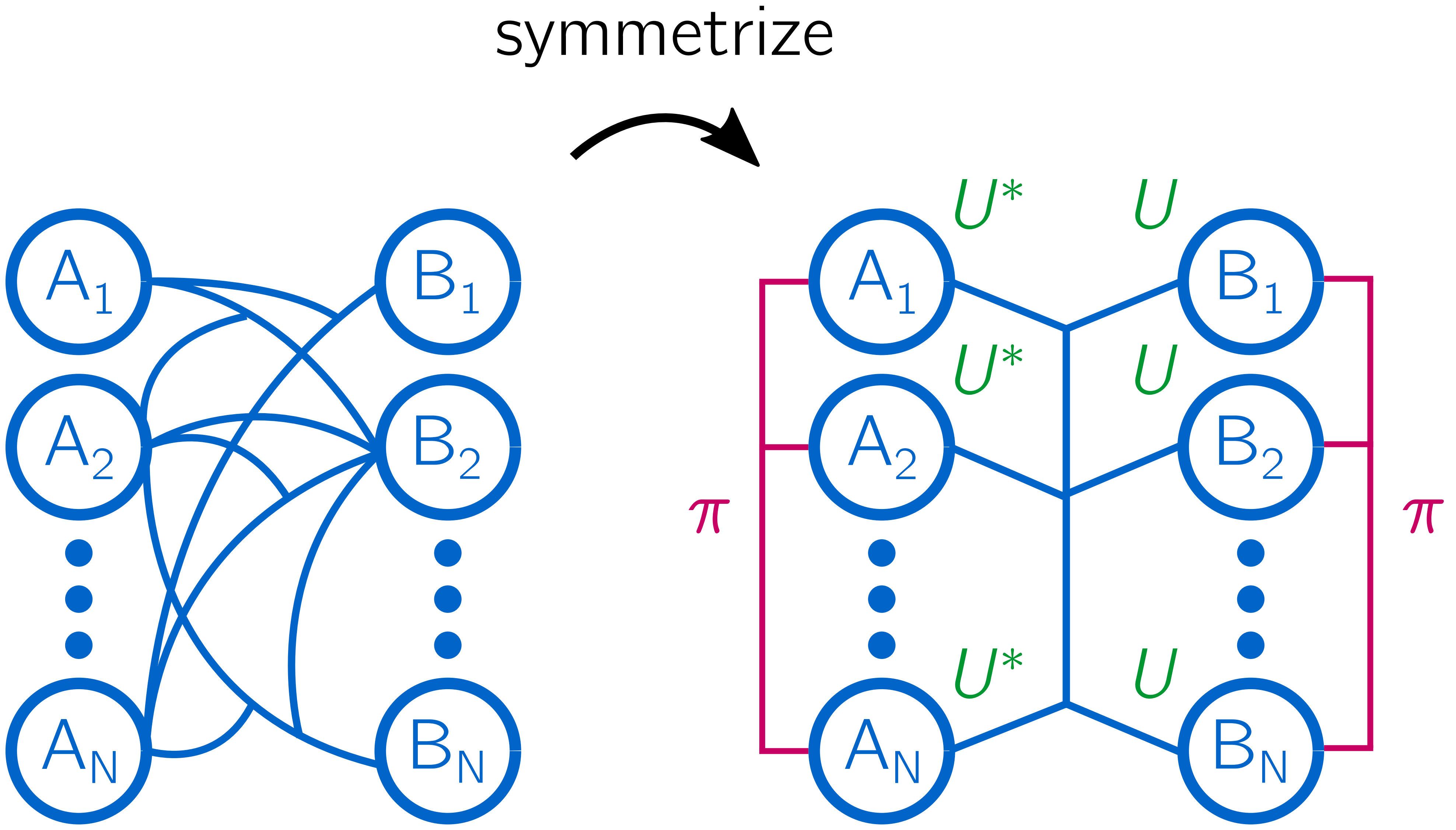
N maximally entangled states Φ^+_{AB} with pretty good measuremnet (optimal!):

asymptotic behavior $F = 1 - O(N^{-1})$.

Better fidelity when optimizing over entangled state $\rho_{A^N B^N}$?

Yes, $F = 1 - \Theta(N^{-2})$.

Arbitrary PBT protocols:
Can always assume $\mathcal{U}_d$ and S_N symmetries as discussed before.



[Majenz '17], [Mozrzymas et al. '18], [arXiv:1809.10751]

Conclusion

Port-based teleportation: approximate teleportation scheme with unitary covariance that enables interesting applications.

Natural symmetries enable characterization of performance using tools from representation theory.

Asymptotics of PBT can be derived using interesting connection between representation theory and random matrix theory.

Can we use these tools to analyze the asymptotic behavior of other quantum-information theoretic tasks with similar symmetries?

Thank you for your attention!