ICTQT/IFTiA/KCIK Seminar

April 10, 2020

Playing Games with Multiple Access Channels

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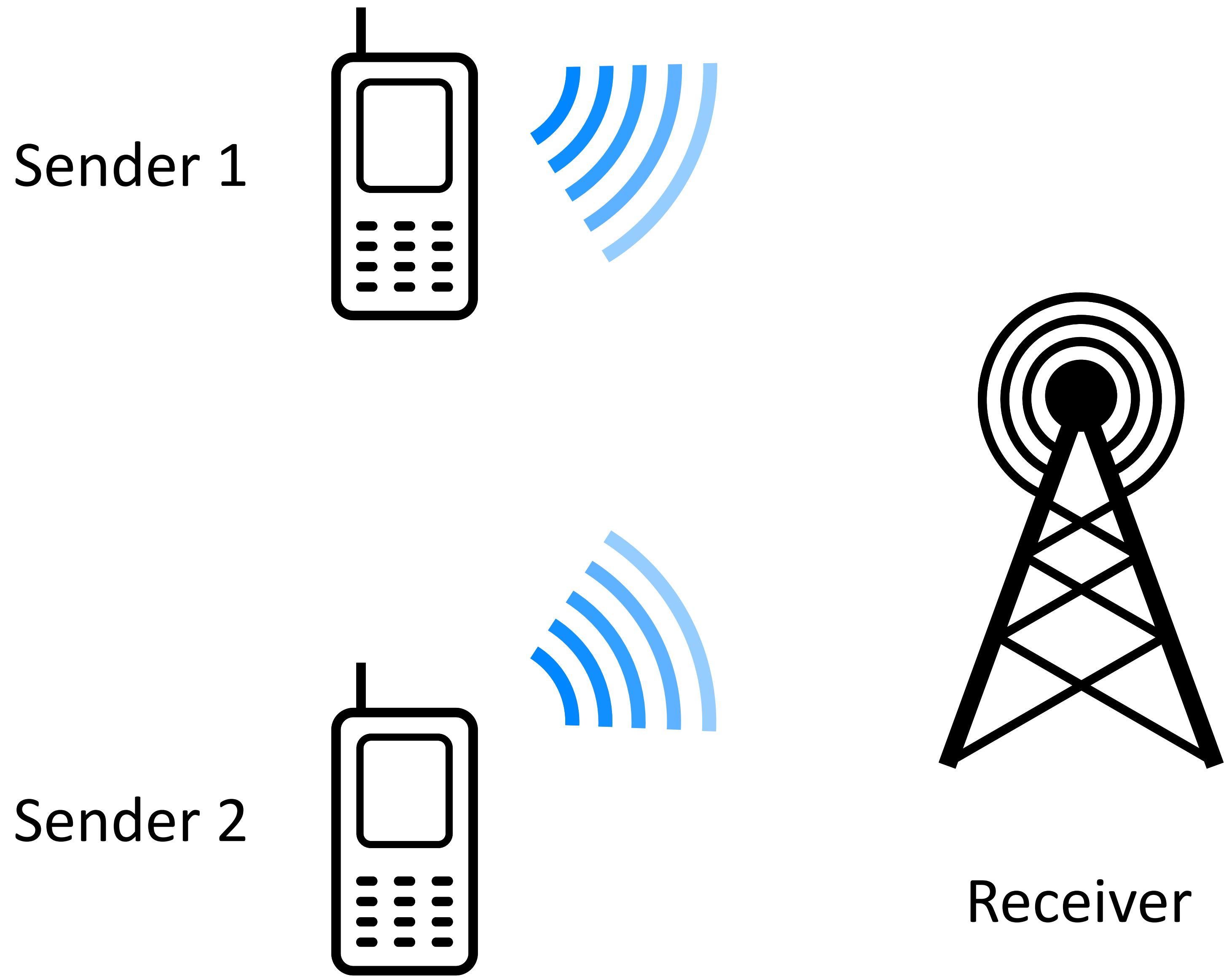
Multiple access channel

Simplest network communication scenario involving two senders and one receiver.



Goal

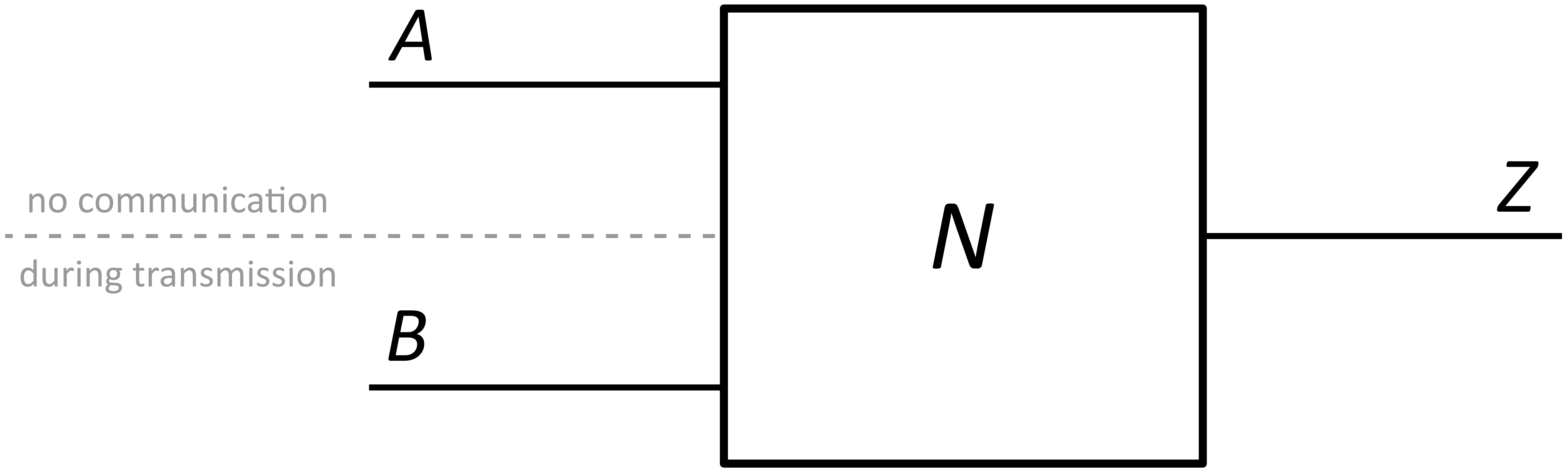
Each sender transmits individual classical messages through common channel to the receiver.

Multiple access channel

MAC: conditional probability distribution $N(z|a, b)$.

Random variables: $(A, B) \xrightarrow{N} Z$

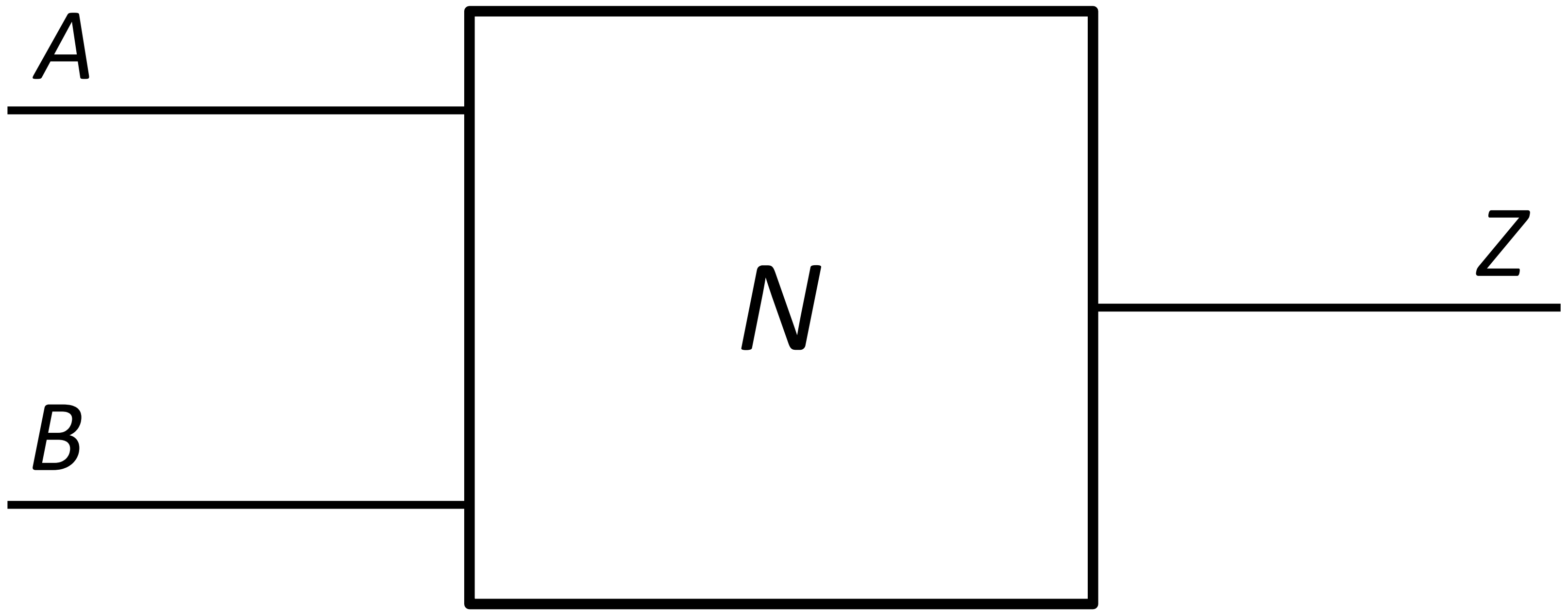
No communication between senders: input RVs A, B are **independent**.



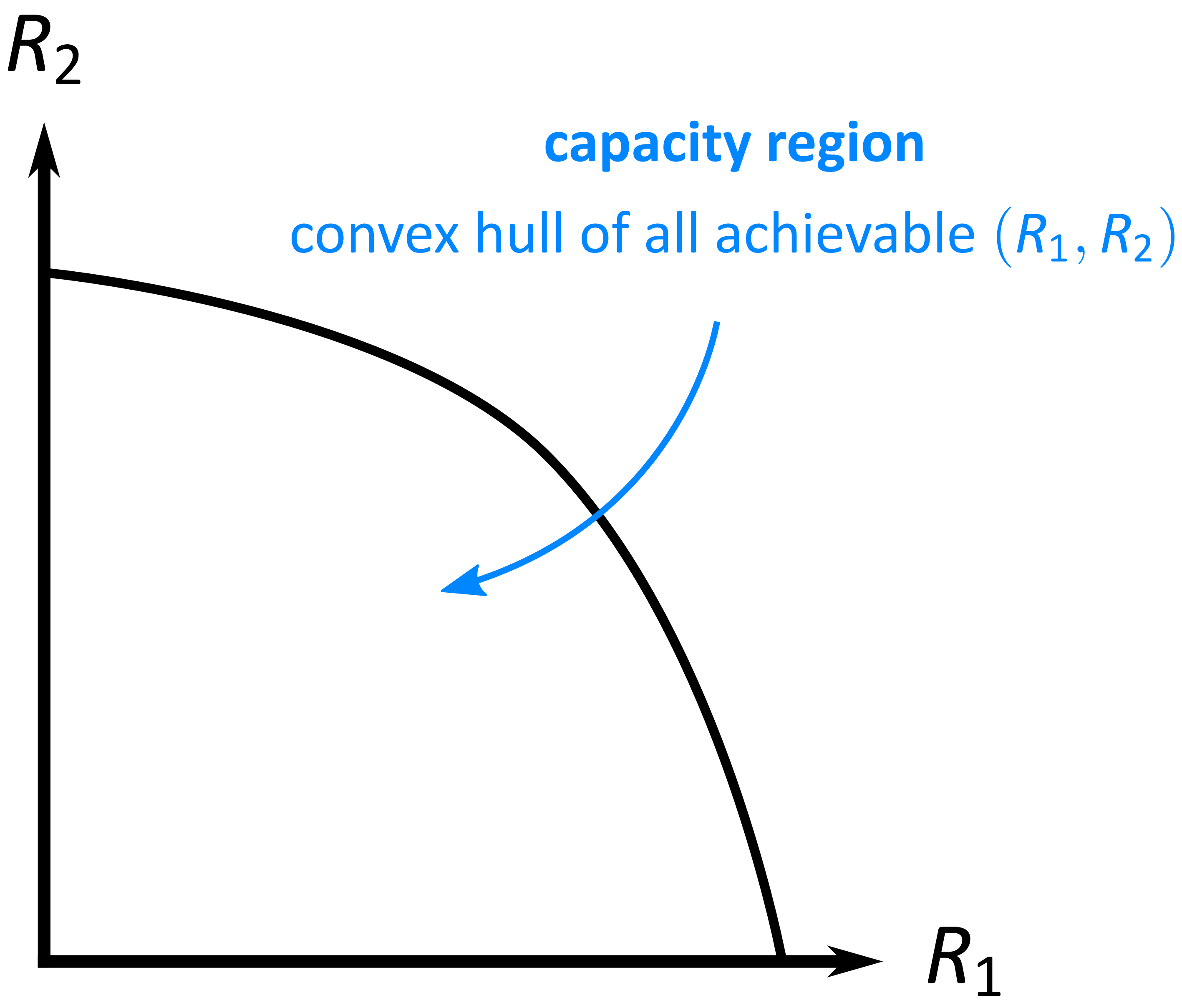
Capacity region of a MAC

Sender 1 (2) tries to send information at rate R_1 (R_2).

(R_1, R_2) is called *achievable* if receiver can decode the two messages with vanishing error.



Multiple access channel



Typical capacity region

Locality & quantum correlations

The independence constraint for the two senders in the MAC scenario can be interpreted as a locality constraint.

Bell inequalities: quantum correlations are strict superset of classical correlations.

Central questions in our work

Can entanglement assistance increase the capacity region of a MAC?

YES (and it can be complicated...)

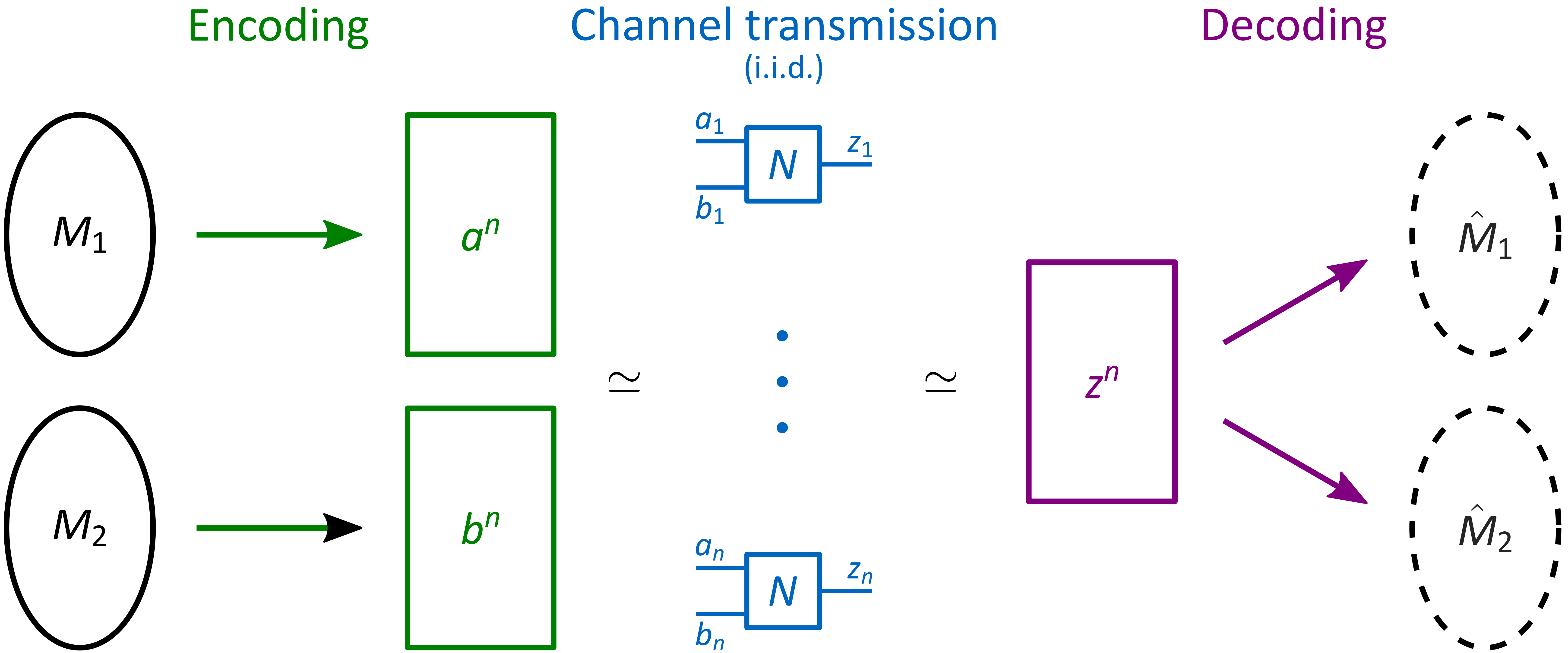
How hard is it to compute the unassisted capacity region of a MAC?

NP-HARD

Talk outline

- Capacity region of a classical MAC and entanglement assistance
- Quantum correlations and non-local games
- Constructing a MAC in terms of a non-local game
- **Main result 1:** entanglement increases capacity region
- **Main result 2:** unbounded entanglement may be necessary
- **Main result 3:** computing the unassisted capacity region is NP-hard
- Conclusion and open questions

Coding for a MAC



Codebooks: $|\mathcal{M}_i| = 2^{nR_i}$

Decoding error: $\Pr((\hat{M}_1, \hat{M}_2) \neq (M_1, M_2))$

Capacity region of a MAC

Decoding error: $\varepsilon_n = \Pr((\hat{M}_1, \hat{M}_2) \neq (M_1, M_2))$

Rate tuple (R_1, R_2) for $R_i = \frac{1}{n} \log |\mathcal{M}_i|$ is called *achievable* if $\varepsilon_n \to 0$ as $n \to \infty$.

Capacity region: $\mathcal{C} := \mathrm{cl}\left(\{(R_1, R_2) \text{ achievable}\}\right)$

Single-letter capacity region of a MAC **(Ahlswede '73, Liao '73)**

Let A and B be RVs with product distribution $p_A(a)p_B(b)$, and Z be a RV defined by the MAC N. Then $\mathcal{C}$ is the convex hull of all (R_1, R_2) with

$$R_1 \leq I(A;Z|B) \qquad R_2 \leq I(B;Z|A) \qquad R_1 + R_2 \leq I(AB;Z).$$

Shannon entropy:

$$H(X) = -\sum_x p(x) \log p(x)$$

Mutual information:

$$I(X;Y) = H(X) + H(Y) - H(XY)$$

Conditional mutual information:

$$I(X;Y|Z) = I(X;YZ) - I(X;Z)$$

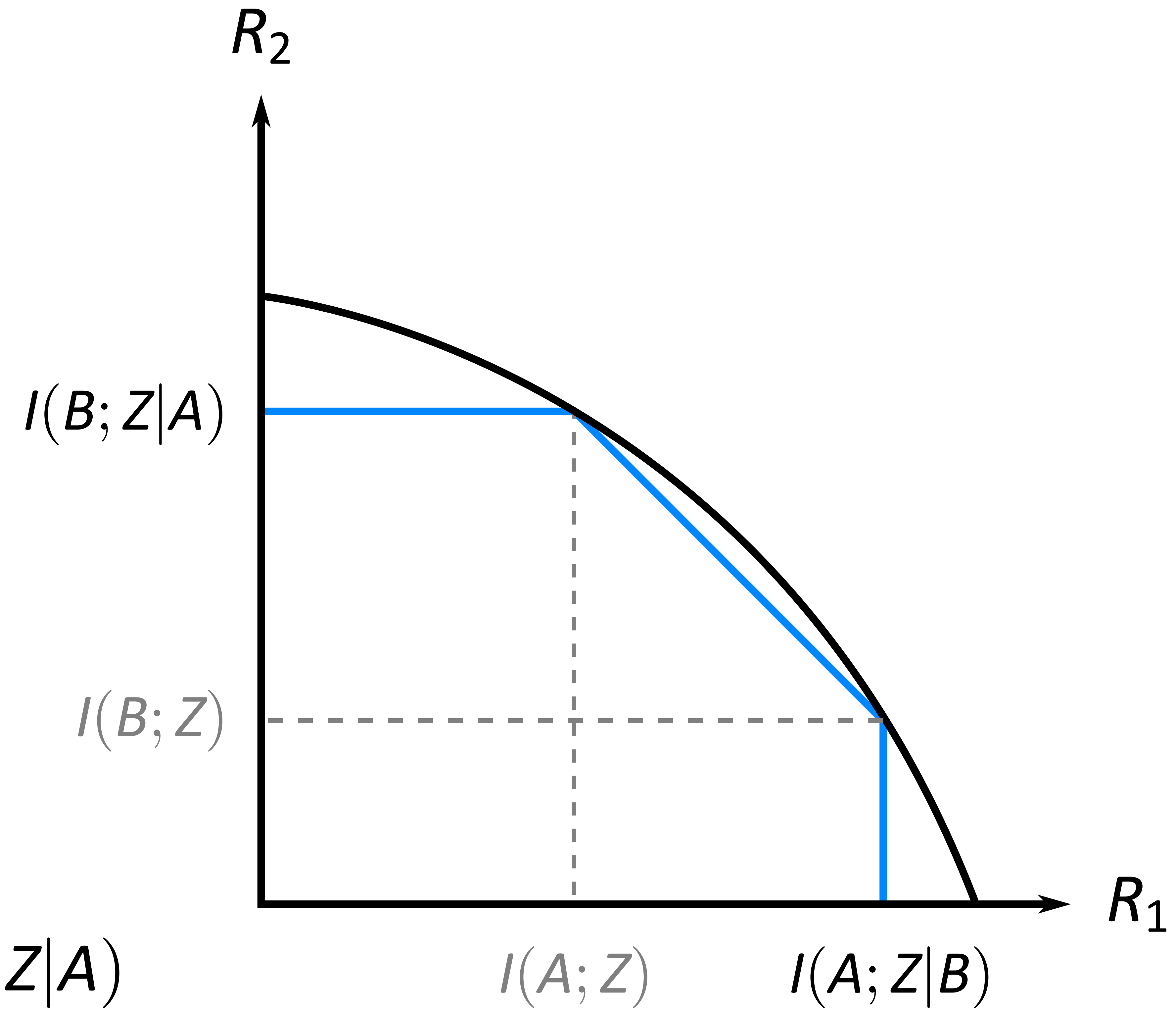
Typical capacity region of a MAC

Constraints for capacity region $\mathcal{C}$:

$$\begin{aligned} R_1 &\leq I(A;Z|B) \\ R_2 &\leq I(B;Z|A) \\ R_1 + R_2 &\leq I(AB;Z). \end{aligned}$$

For fixed product distribution $p_A p_B$ this region is **pentagonal**, since:

$$\begin{aligned} \max\{I(A;Z|B), I(B;Z|A)\} &\leq I(AB;Z) \\ &\leq I(A;Z|B) + I(B;Z|A) \end{aligned}$$



Capacity region of a MAC

Ahlswede-Liao region characterized by **single-letter formula**.

Complicated part: **product constraint** (←independence constraint) on input RVs.

Question 1

Can we use **entanglement assistance** to overcome independence constraint?

Question 2

How hard is it to compute the full region? Product constraint can be turned into **rank-1 constraint.**

[Calvo et al., IEEE Trans. Comm. 58.12 (2010)]

We will study both questions using the theory of **non-local games**.

For simplicity: focus on the **sum rate** $\max\{R_1 + R_2 : (R_1, R_2) \in \mathcal{C}(N)\}$.

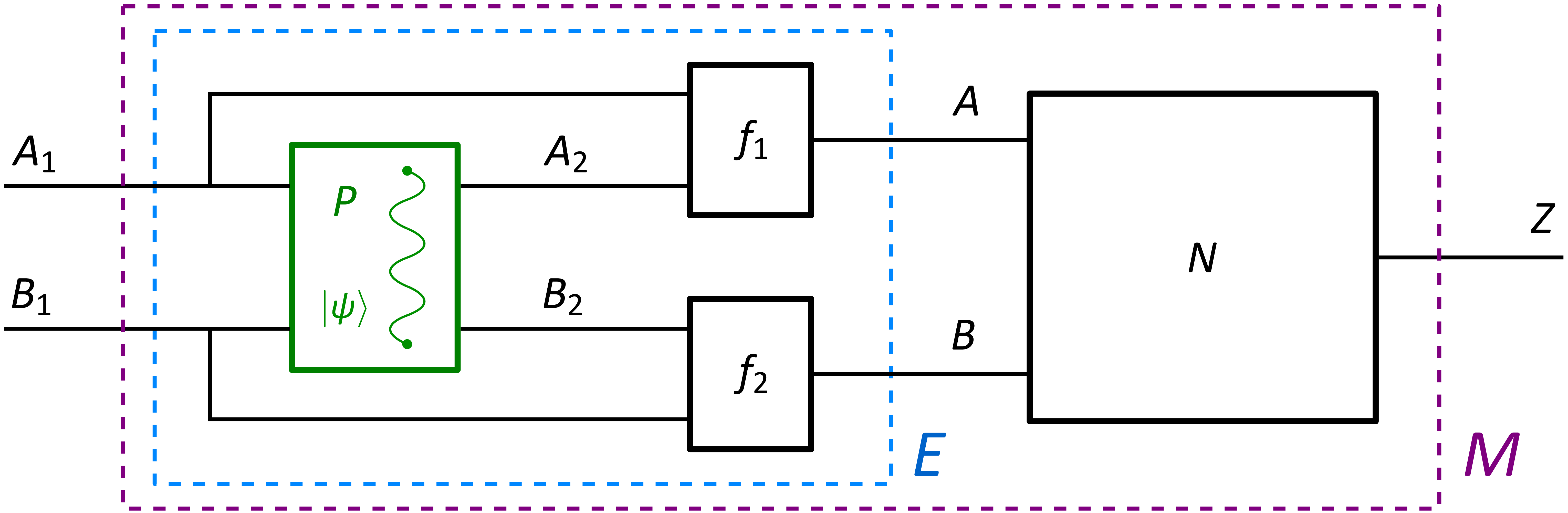
Entanglement assistance for MACs

Senders share entangled state $|\psi\rangle$ and POVMs $\{\Pi_{a_2}^{a_1}\}_{a_2}$ and $\{\Pi_{b_2}^{b_1}\}_{b_2}$:

$$P(a_2, b_2|a_1, b_1) = \langle\psi|\Pi_{a_2}^{a_1} \otimes \Pi_{b_2}^{b_1}|\psi\rangle.$$

Resulting correlation: $E(a, b|a_1, b_1) = f_1(a|a_1, a_2)f_2(b|b_1, b_2)P(a_2, b_2|a_1, b_1)$.

Total MAC: $M = N \circ E$



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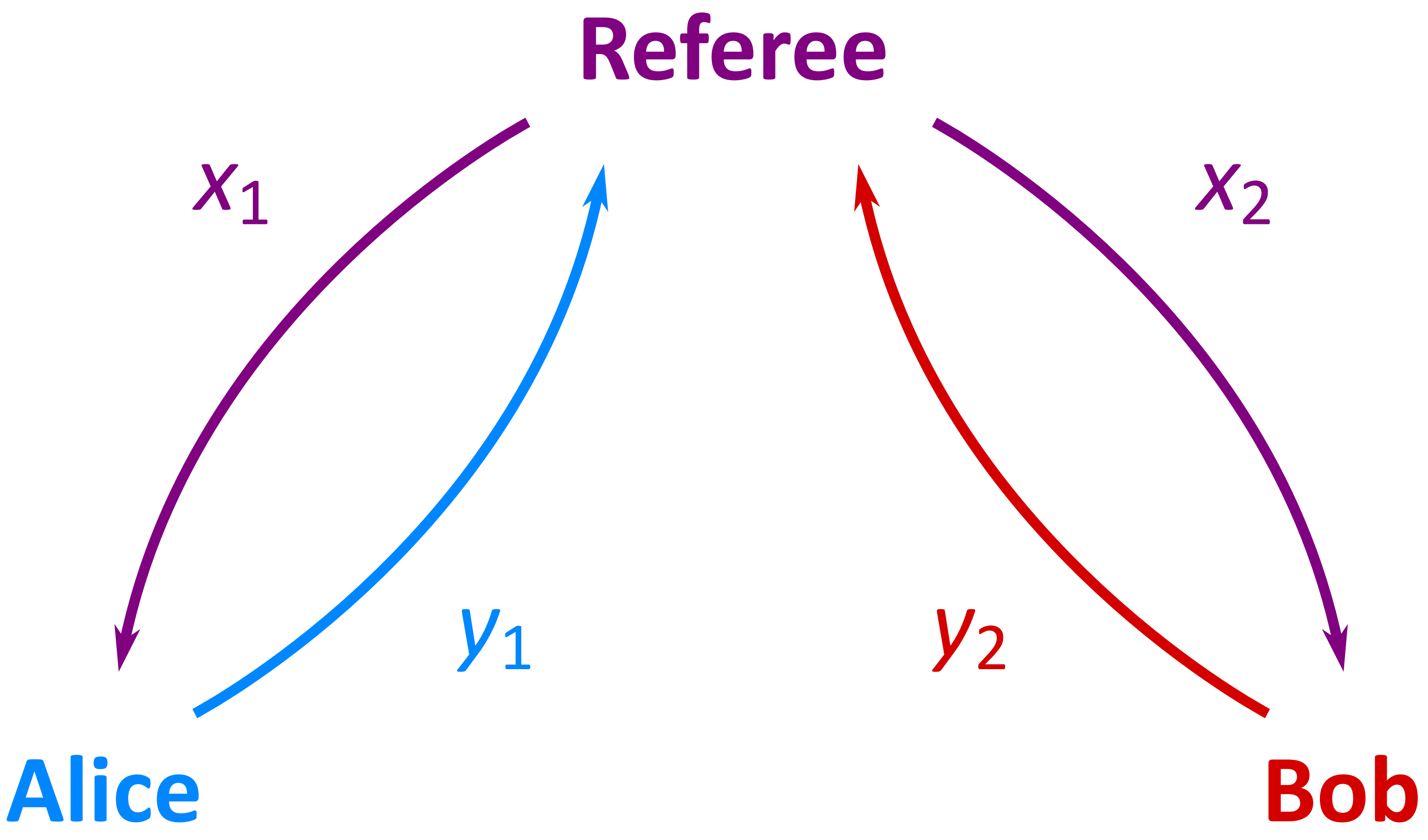
Non-local games



Questions $x_i \in \mathcal{X}_i$

Answers $y_i \in \mathcal{Y}_i$

Winning condition $W \subseteq \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}_1 \times \mathcal{Y}_2$

Non-local game $G = (\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2, W)$.

Referee draws questions x_i according to some distribution.

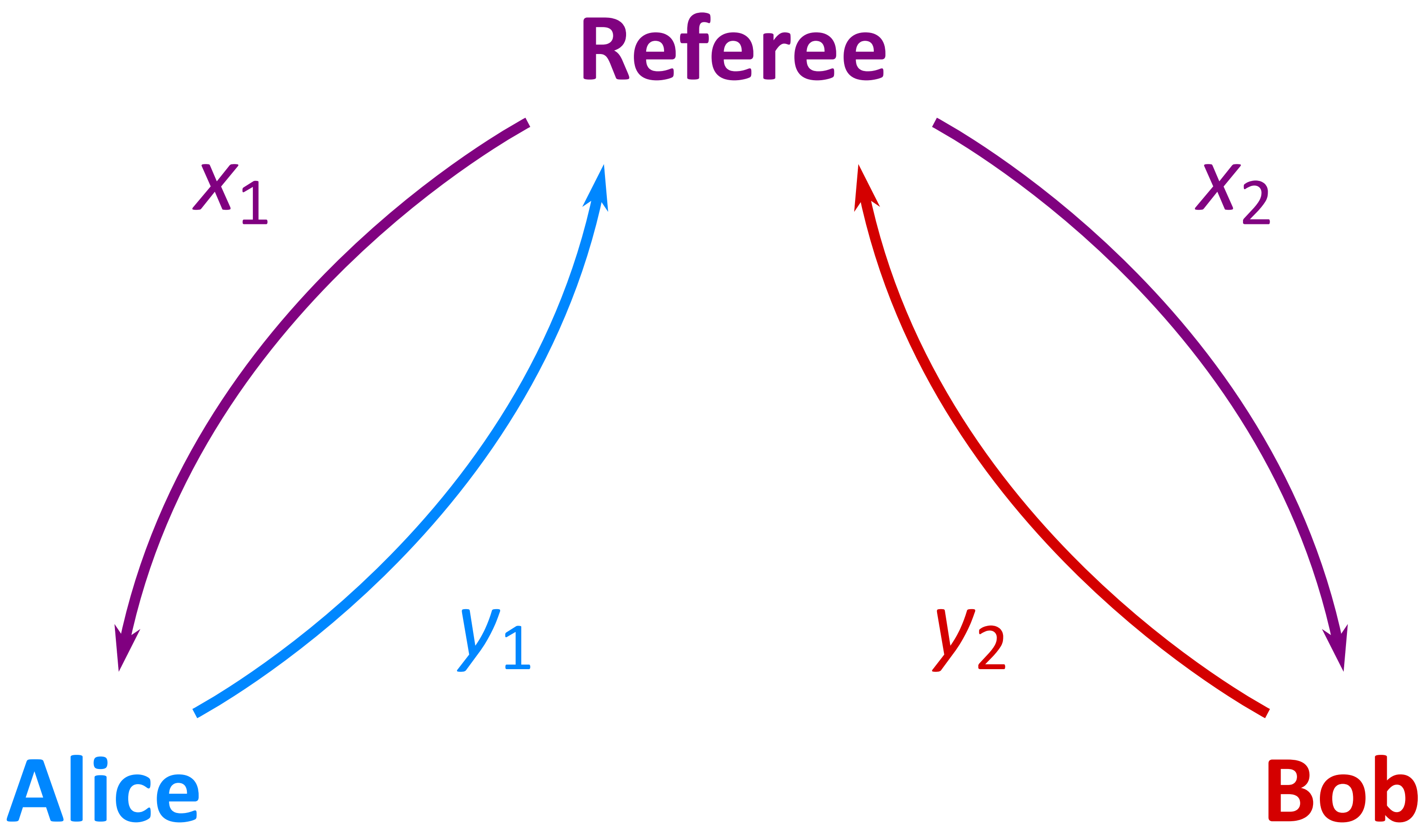
No communication allowed for Alice and Bob to produce answers y_i.

Alice and Bob win if $(x_1, y_1, x_2, y_2) \in W$.

Example: CHSH game

Winning condition: $y_1 \oplus y_2 = x_1 \wedge x_2$

Non-local games: Classical strategies



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Deterministic strategy:

Deterministic functions $f_i \colon \mathcal{X}_i \to \mathcal{Y}_i$.

Probabilistic strategy:

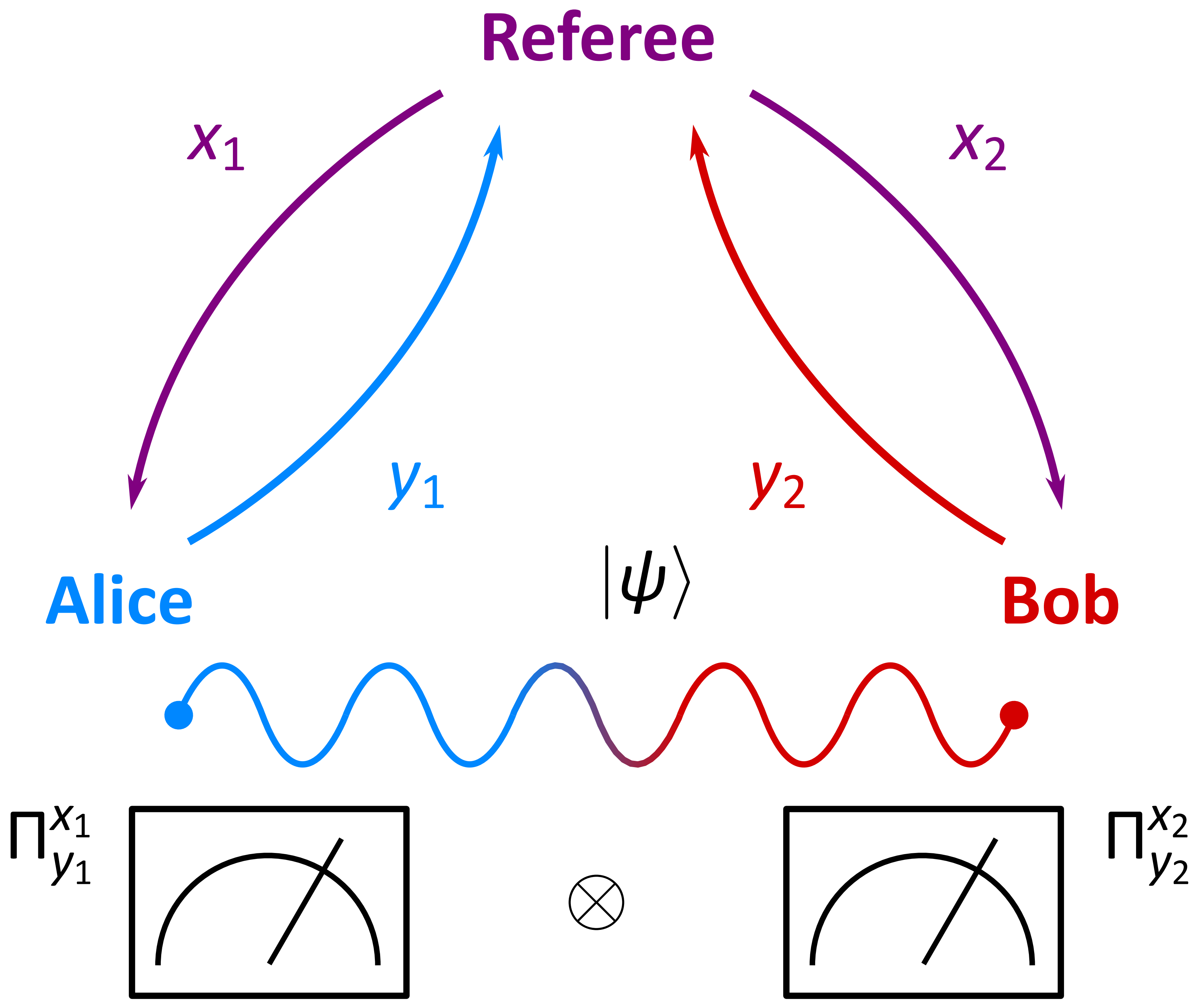
Probabilistic mixture of deterministic strategies.

Classical value $\omega(G)$:

Maximal classical winning probability.

$\omega(G)$ depends on distribution on questions (x_1, x_2).

Non-local games: Quantum strategies



$\Pi^{x_i}_{y_i} \geq 0, \quad \sum_{y_i} \Pi^{x_i}_{y_i} = I$

Quantum strategies:

Alice and Bob share **entangled state** $|\psi\rangle$.

Select **POVMs** $\{\Pi^{x_i}_{y_i}\}_{y_i \in \mathcal{Y}_i}$ for each $x_i \in \mathcal{X}_i$

$(x_1, x_2) \mapsto (y_1, y_2)$ w.p. $\langle\psi|\Pi^{x_1}_{y_1} \otimes \Pi^{x_2}_{y_2}|\psi\rangle$.

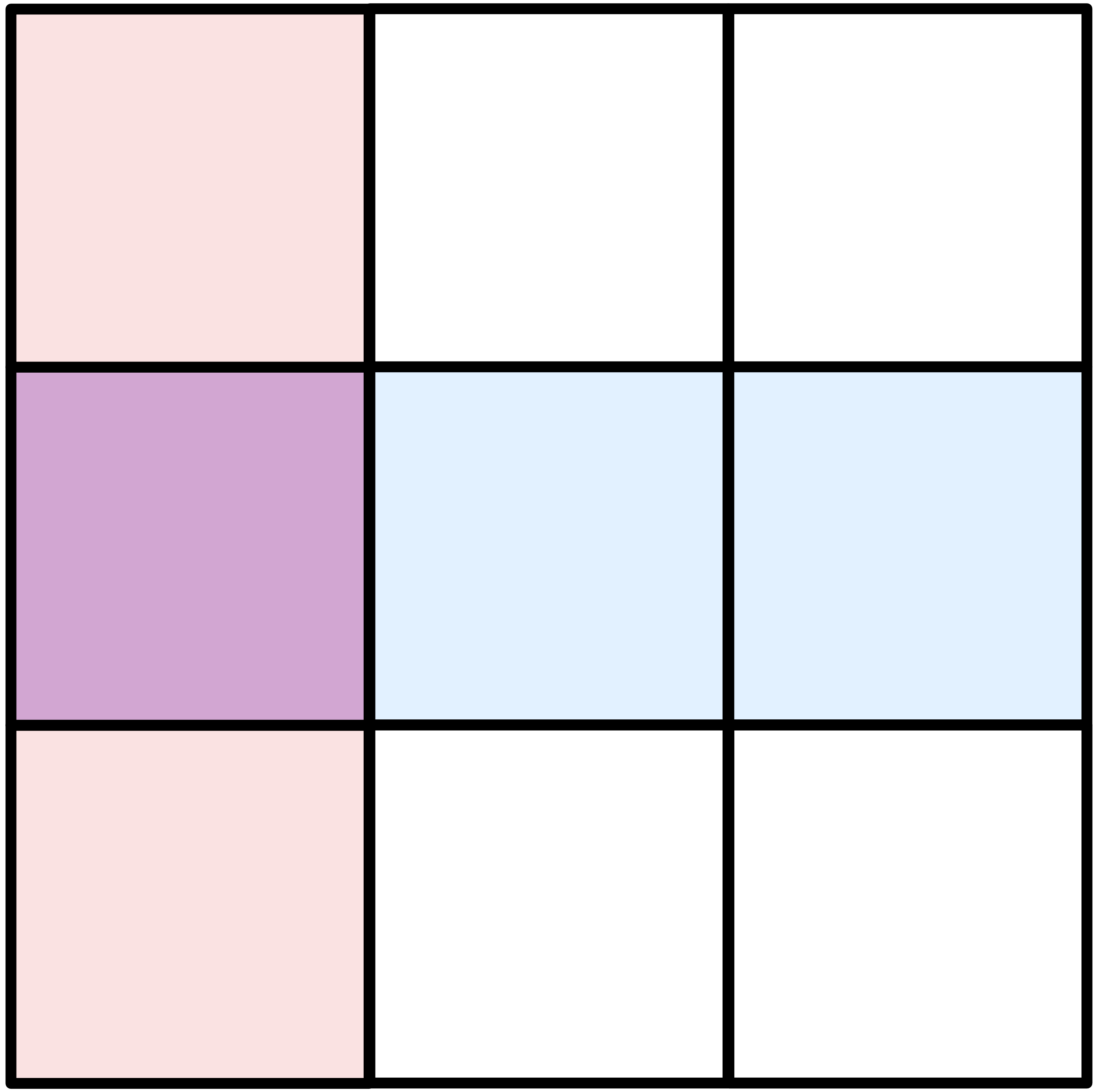
Quantum value $\omega^*(G)$:

maximal quantum winning probability.

Example: CHSH-game G_C

$0.75 = \omega(G_C) < \omega^*(G_C) \approx 0.85$

Magic square game



Alice is given a row.

Bob is given a column.

Both answer with strings of length 3.

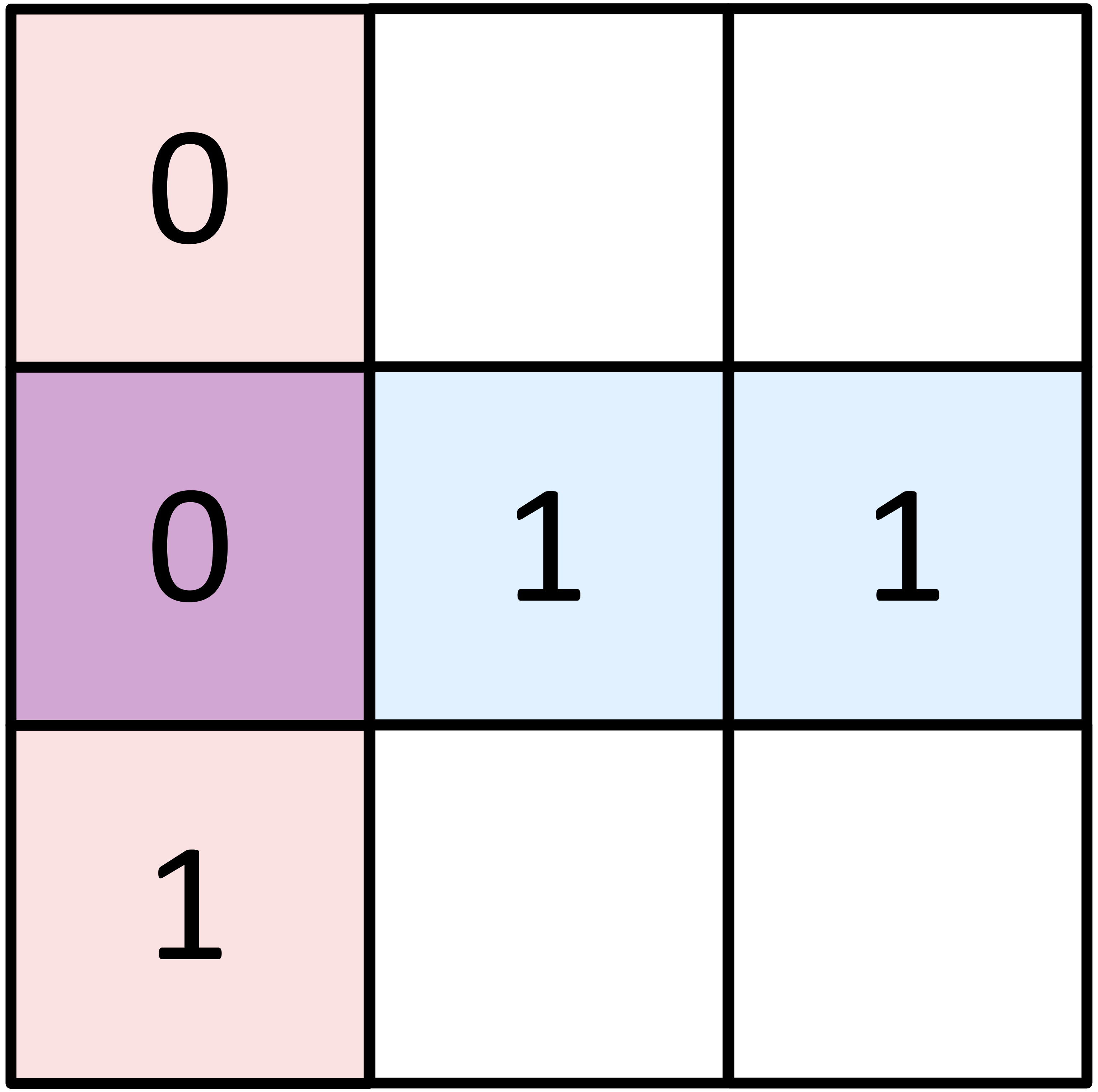
They win, if:

- Alice's parity is even;
- Bob's parity is odd;
- strings agree in overlapping cell.

[Mermin, PRL 65.27 (1990)]

[Peres, Phys. Lett. A 151.3 (1990)]

Magic square game



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Magic square game

0	0	0
0	1	1
1	0	?

Alice is given a row.

Bob is given a column.

Both answer with strings of length 3.

They win, if:

- Alice's parity is even;
- Bob's parity is odd;
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MSG: Classical strategies

0	0	0
0	1	1
1	0	?

Perfect deterministic strategy **necessarily violates** parity constraints.

Maximal winning probability: 8/9

For uniformly drawn questions, this also holds for any probabilistic strategy.

Classical value

$\omega(G_{MS}) = 8/9$

[Brassard et al., Found. Phys. 35.11 (2005)]

MSG: A perfect quantum strategy

+XI	+XX	+IX
-XZ	+YY	-ZX
+IZ	+ZZ	+ZI

Let Alice and Bob share two EPR pairs $|\Phi\rangle_{A_1B_1}|\Phi\rangle_{A_2B_2}$, and measure the observables in their row/column.

Observables **commute along rows and columns**.

Parity constraints are **always satisfied**.

Quantum value

$\omega^*(G_{MS}) = 1$

[Mermin, PRL 65.27 (1990)], [Peres, Phys. Lett. A 151.3 (1990)]

[Brassard et al., Found. Phys. 35.11 (2005)]

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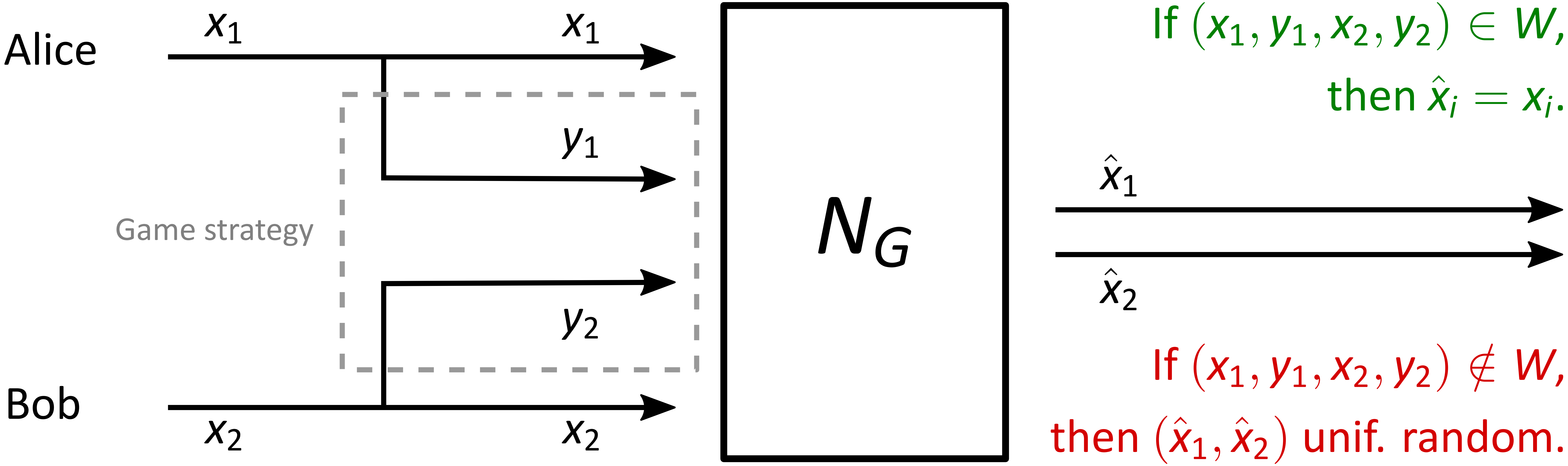
MAC in terms of a non-local game

Let $G = (\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2, W)$ be a non-local game.

Inputs: question-answer pair (x_i, y_i)

Output: question pair $(\hat{x}_1, \hat{x}_2)$



If $(x_1, y_1, x_2, y_2) \in W$, then $\hat{x}_i = x_i$.

If $(x_1, y_1, x_2, y_2) \notin W$, then $(\hat{x}_1, \hat{x}_2)$ unif. random.

Inspired by [Quek & Shor, PRA 95.5 (2017)].

MAC in terms of a non-local game

Let $G = (\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2, W)$ be a non-local game.

$$N_G(\hat{x}_1, \hat{x}_2 | x_1, y_1, x_2, y_2) = \begin{cases} \delta(\hat{x}_1, x_1)\delta(\hat{x}_2, x_2) & \text{if } (x_1, y_1, x_2, y_2) \in W \\ (|\mathcal{X}_1||\mathcal{X}_2|)^{-1} & \text{else.} \end{cases}$$

Operational connection to the actual non-local game G:

Alice and Bob ask themselves x_i independently, then produce y_i using a game strategy.

$$\pi(x_1, y_1, x_2, y_2) = \pi(x_1)\pi(x_2)\pi(y_1, y_2 | x_1, x_2)$$

Probabilistic strategies:

$$\pi(y_1, y_2 | x_1, x_2) = \sum_\lambda \pi_\lambda f_1(y_1 | x_1, \lambda) f_2(y_2 | x_2, \lambda)$$

Quantum strategies:

$$\pi(y_1, y_2 | x_1, x_2) = \langle \psi | \Pi_{y_1}^{x_1} \otimes \Pi_{y_2}^{x_2} | \psi \rangle$$

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Sum rate of a non-local game MAC

Let $G = (\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2, W)$ be a non-local game and N_G the MAC derived from it.

Lemma

Let $p_L = \Pr\{(x_1, y_1, x_2, y_2) \notin W)\}$ be the **losing probability**, and set $Z = (\hat{X}_1, \hat{X}_2)$.

Then $R_1 + R_2 \leq I(X_1 Y_1 X_2 Y_2; Z) = H(Z) - p_L(\log|\mathcal{X}_1| + \log|\mathcal{X}_2|)$.

RHS is maximal when:

1) $H(Z) = \log|\mathcal{X}_1| + \log|\mathcal{X}_2|$;
   only possible with sampling x_i uniformly at random!
2) $p_L = 0$.

Problem

For a non-local game G with classical value $\omega(G) < 1$ players **cannot** win on all questions!

Sum rate of a non-local game MAC

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Then $R_1 + R_2 \leq I(X_1 Y_1 X_2 Y_2; Z) = H(Z) - p_L(\log|\mathcal{X}_1| + \log|\mathcal{X}_2|)$.

Main result: No-Go theorem for classical strategies

For a non-local game with classical value $\omega(G) < 1$,

$$R_1 + R_2 < \log|\mathcal{X}_1| + \log|\mathcal{X}_2|.$$

Sum rate of a non-local game MAC

Let $G = (\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2, W)$ be a non-local game and N_G the MAC derived from it.

Lemma

Let $p_L = \Pr\{(x_1, y_1, x_2, y_2) \notin W)\}$ be the **losing probability**, and set $Z = (\hat{X}_1, \hat{X}_2)$.

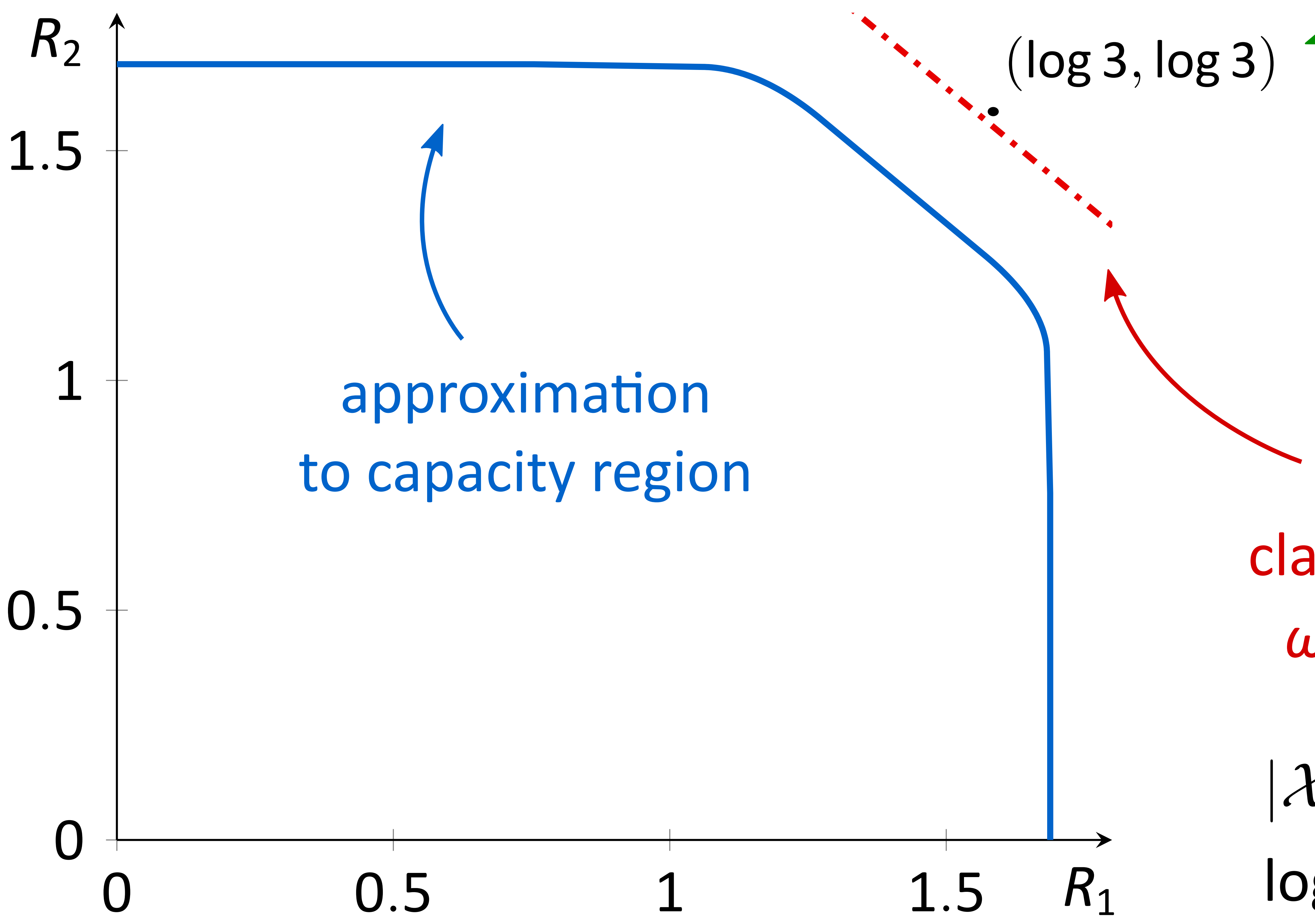
Then $R_1 + R_2 \leq I(X_1 Y_1 X_2 Y_2; Z) = H(Z) - p_L(\log|\mathcal{X}_1| + \log|\mathcal{X}_2|)$.

Main result: perfect sum rate with entanglement

If $\omega^*(G) = 1$, then the **perfect** quantum strategy can be used to **achieve** $(R_1, R_2) = (\log|\mathcal{X}_1|, \log|\mathcal{X}_2|)$ by drawing (x_1, x_2) uniformly at random.

$$\Rightarrow R_1 + R_2 = \log|\mathcal{X}_1| + \log|\mathcal{X}_2|$$

Example: Magic square game channel



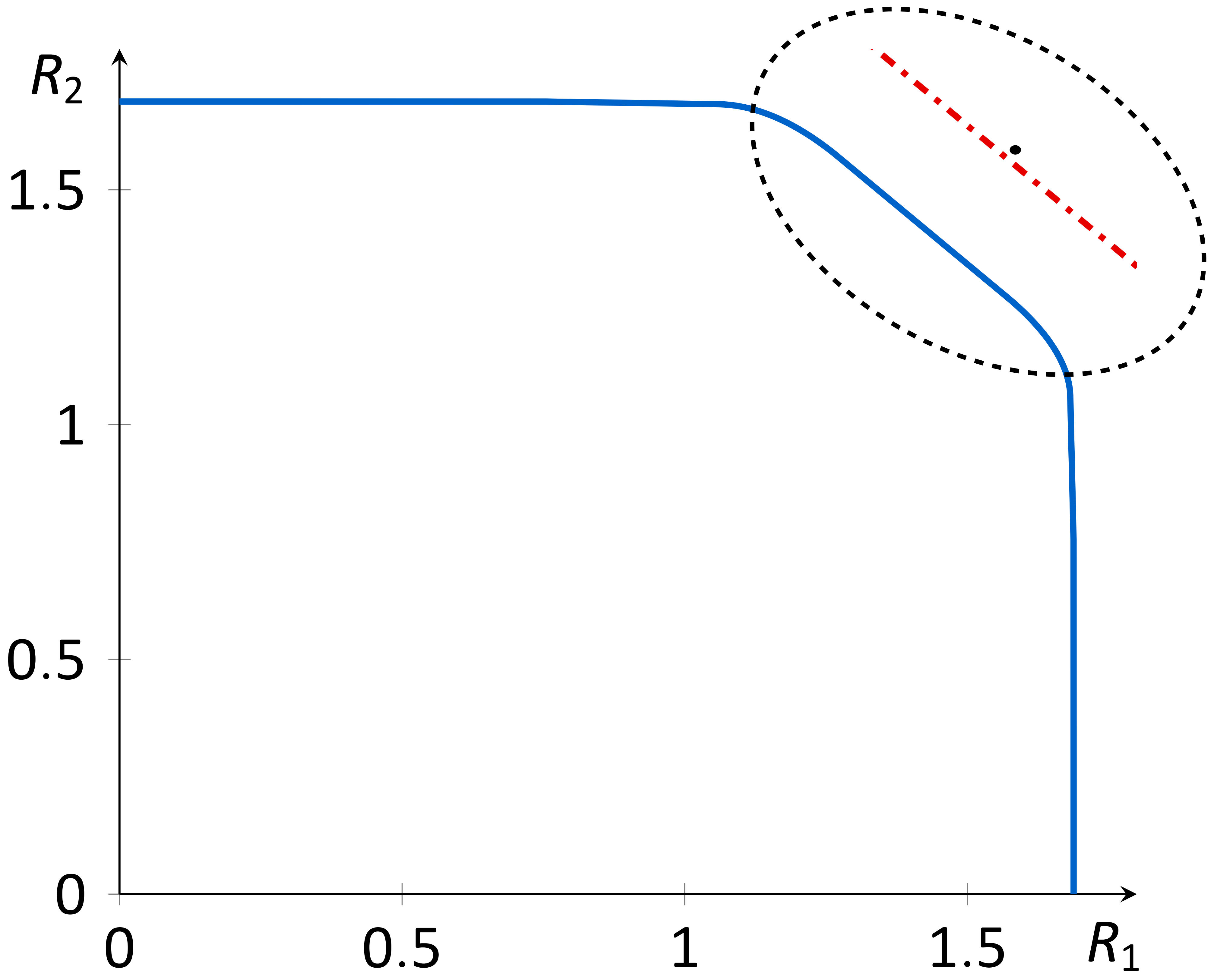
achievable using
perfect quantum strategy
$\omega^*(G_{MS}) = 1$

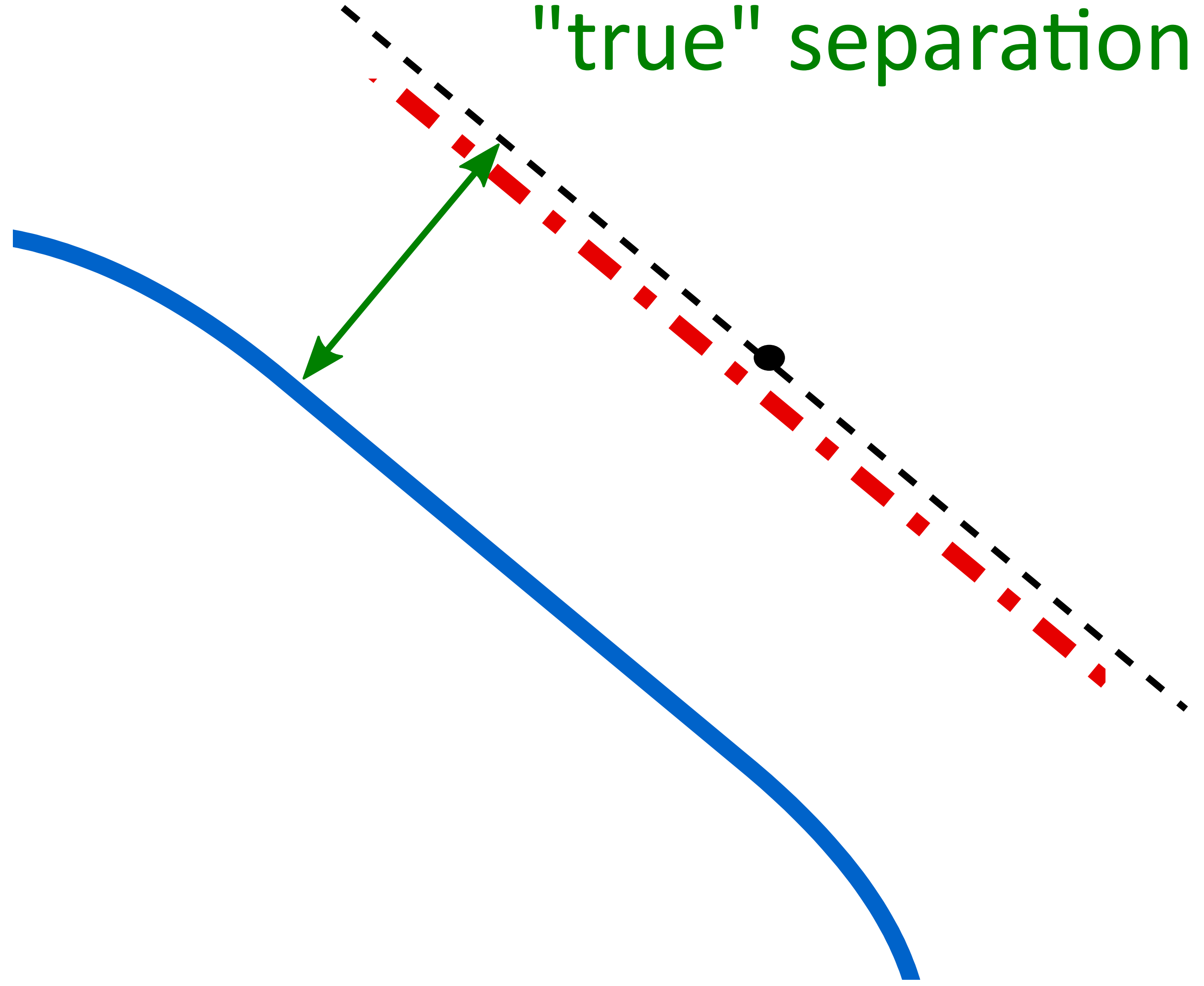
Bound on
classical sum rate
$\omega(G_{MS}) = 8/9$

$|\mathcal{X}_1| = |\mathcal{X}_2| = 3, |\mathcal{Y}_1| = |\mathcal{Y}_2| = 8$

$\log 3 \approx 1.585$

Example: Magic square game channel





Separation from bound: 0.033

“True” separation: 0.328

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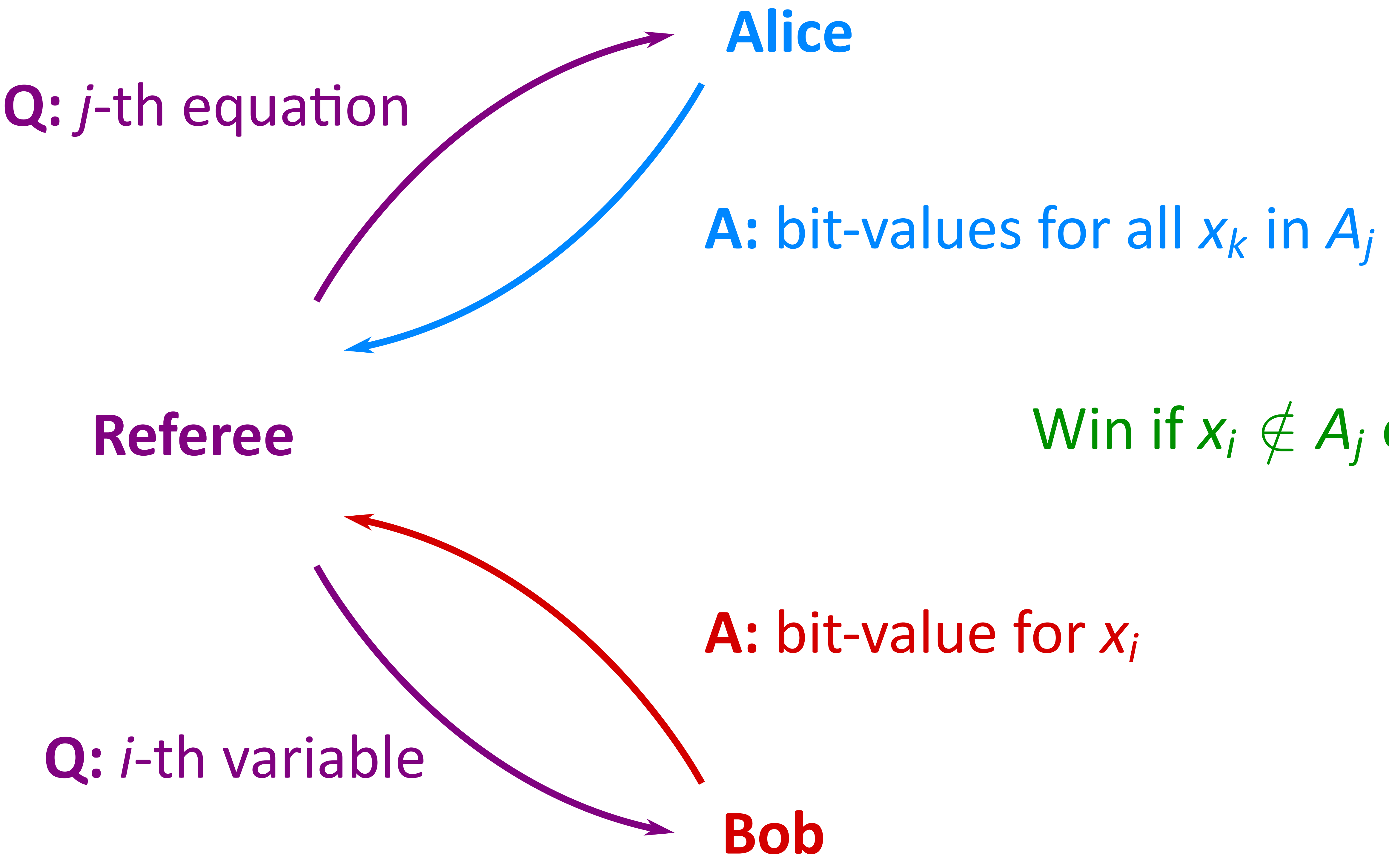
Linear system games

Given: Linear system of m equations $Ax = b$ in n variables x_i over $\mathbb{F}_2$.



Win if $x_i \notin A_j$ or answers consistent.

Unbounded entanglement needed

$\exists\, G_{SV} = (A, b)$ such that $\omega^*(G_{SV}) < 1$ for **any finite-dimensional entangled strategy**.

For MES with Schmidt rank d: $\frac{C_1}{d^6} \leq p_L \leq \frac{C_2}{d^2} \Rightarrow$ $\omega^*(G_{SV}) < 1$ for $d < \infty$; $\omega^*(G_{SV}) \to 1$ for $d \to \infty$

Main result: Unbounded entanglement

Any d-entangled strategy for the MAC $N_{G_{SV}}$ must have $R_1 + R_2 < \log m + \log n$.

There is an entangled strategy such that $R_1 + R_2 \to \log m + \log n$ as $d \to \infty$.

For the family of *all* linear system games, it is **undecidable** whether $(\log m, \log n)$ can be achieved for the corresponding family of MACs.

[Slofstra and Vidick, Ann. H. Poincare 19.10 (2018)], [Slofstra, Forum Math. Pi 7 (2019)]

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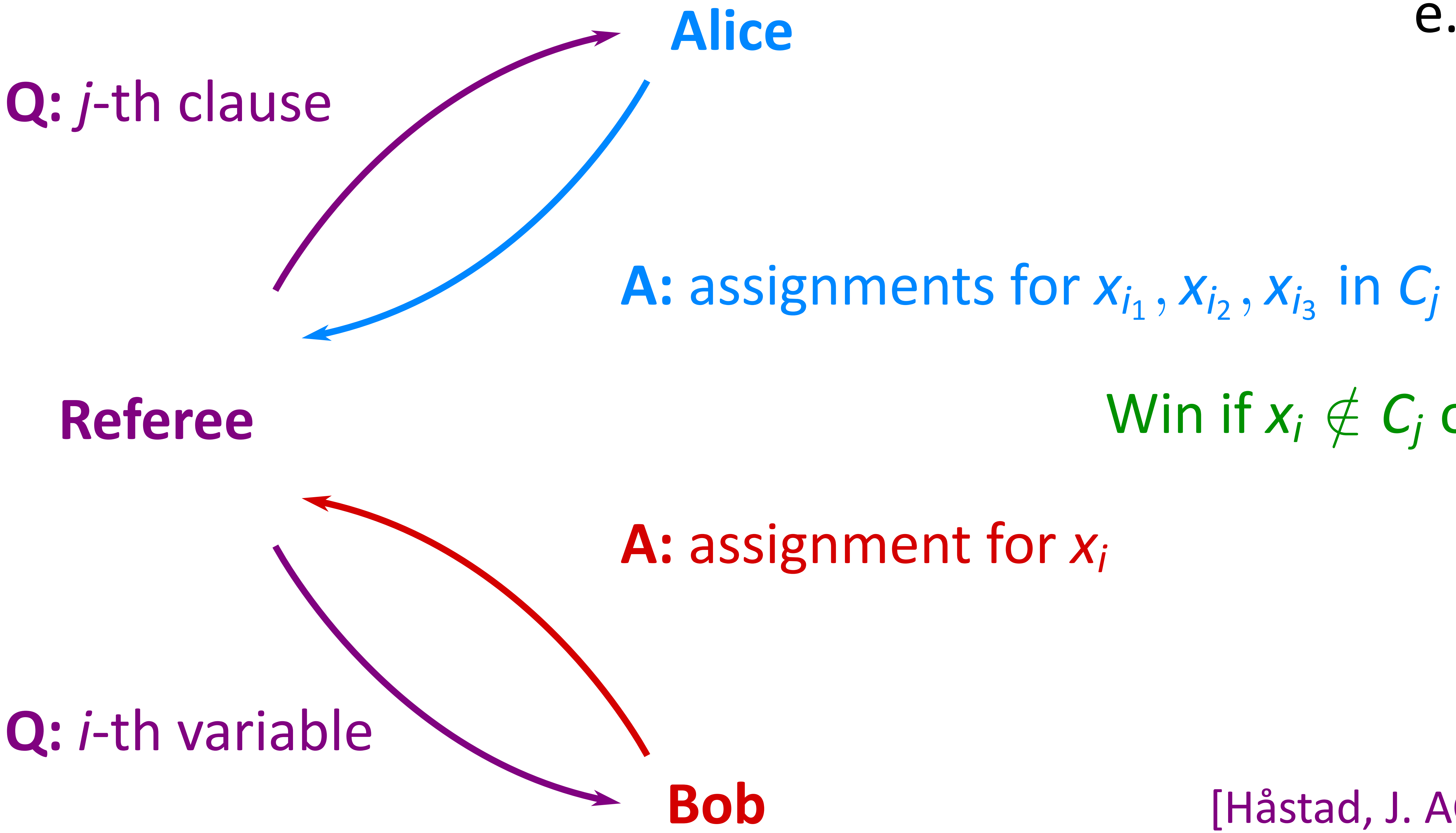
A non-local game version of 3-SAT

Given: Boolean variables $x_1, \ldots, x_n$ and $C_1, \ldots, C_m$ clauses containing exactly 3 literals,

e.g. $C_i = x_2 \vee \neg x_4 \vee x_5$.



Win if $x_i \notin C_j$ or answers consistent.

[Håstad, J. ACM 48.4 (2001)]

NP-hardness of computing capacity region

PCP Theorem: It is NP-hard to decide for Håstad's game G_H with $m = O(n)$ whether $\omega(G_H) = 1$ or $\omega(G_H) \leq 1 - (1-c)/n$ for some $c < 1$.

Main result: NP-hardness of computing unassisted capacity region

For MAC N_{G_H}, it is NP-hard to decide whether $R_1 + R_2 = \log m + \log n$ can be achieved or $R_1 + R_2 \leq \log m + \log n - ((1-c)/n)^3$.

For a point-to-point channel with $O(n)$ bit inputs, we can approximate capacity to precision $O(n^{-3})$ in time $O(n^3 \log n)$ using Blahut-Arimoto algorithm.

With common assumptions about 3-SAT, the scaling for a MAC is $\exp(O(n))$.

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Conclusion

MAC models simple network communication scenario with 2 senders, 1 receiver.

Capacity region given by **single-letter** formula, but **non-convex** problem.

Main results

- Entanglement between senders can boost capacity region of a MAC.
- You may need lots of entanglement to get full boost.
- This is generally undecidable.
- The classical capacity region is NP-hard to compute.

All results are proven by embedding a non-local game in a MAC scenario.

Open questions

Information-theoretic

- Can we improve sum rate bound to get "true" separation?
- Formula for the entanglement-assisted capacity region?
- What about arbitrary (three-way) entanglement assistance?

Optimization-theoretic

- Efficiently computable outer bounds for capacity region of MAC?
- Efficient optimization over (bilinear) quantum strategies?
- Can entanglement boost the capacity of arbitrary MACs?

Thank you

for your attention!